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AN ANALYTICAL MODEL OF A HORIZONTAL AXIS WIND TURBINE BLADE WITH COUPLED FLAPWISE-EDGEWISE-TORSION MOTIONS

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Abstract**—**Aeronautical and aerospace engineering's current push towards lightweight structures is producing airfoils with slender designs, which are more prone to experience geometrical nonlinearities due to large deformation. In this paper, Hamilton's principle is used to derive the bending-torsion coupling equations. The model incorporates the blades' edgewise, flapwise, and torsion deformations. The nonlinear vibration is described by a system of partial differential equations. Vibrations of the blade are theoretically analyzed to determine the natural frequencies. Amplitudes of vibration and natural frequencies of various blade modes obtained in the present study are compared with those previously obtained. The model proposed in this paper can be widely used in the future study.

Keywords – coupled vibrations; natural frequency; multiple-scales method; internal resonance

I. INTRODUCTION

Reliance on fossil fuel resources must be replaced with alternative, yet environmentally friendly, renewable energy sources due to the depletion and environmental effects of fossil fuels as well as the growing demand for energy consumption. Among all renewable energy sources, wind energy is one of the most popular and rapidly growing. The technological advancements have led to the development of wind turbines with greater capacity and structural scale. To capture more wind energy, blade length is increased. To evaluate and understand the complex interaction between the wind turbines' elastic vibrations and the unsteady aerodynamic forces acting upon them, a thorough analysis of the system dynamics is necessary as blades becoming lighter and more flexible.

Vibration of a wind turbine blade is usually studied in four directions namely, axial, in-plane (edgewsie), out-of-plane (flapwise) and torsion motions. Numerous studies have focused on enhancing computational models to produce accurate outcomes. To determine the natural frequencies of wind turbine blades, Park et al. [1] suggested a formulation with a rigid hub and flexible blade. Wang et al. [2] computed natural frequencies and tip displacements using a mathematical model, based on the Timoshenko beam theory and thin-walled structure theory. Using the differential quadrature approach, Hsu [3] investigated the vibration characteristics of wind turbine blade and studied the effects of pitch angle, rotational speed, and sample count on natural frequencies. Using the Floquet method, Stol et al. [4] studied the effects of precone angle, teeter stiffness, yaw stiffness, teeter damping, and yaw damping on the frequencies for two-bladed wind turbine's operating modes. Gangele and Ahmed [5] examined the dynamic characteristics of a 1.5 MW wind turbine system considering the influence of material and geometrical parameters. The Timoshenko beam for coupled bending torsion vibrations was modeled by Bercin and Tanaka [6]. They have included the warping effects, rotating inertia, and shear deformations into the coupled equations of motion. Bishop and Price [7] studied the coupled bending-torsion vibrations and have considered the effects of rotary inertia and shear deformation. Subrahmanyam et al. [8] modeled the blade as Timoshenko beam and examined the bending and torsion coupling of rotating blades having asymmetric cross section. Kumar et al. [9] studied the dynamic behaviour of an Al 2024 wind turbine blade using the finite element (FE) method.

This study extends the partial differential equations of blade motion proposed by Saram and Yang [10] by considering the torsion motion in the coupled flapwise-edgewise-torsion vibrations. To investigate the natural frequencies and amplitudes of vibration of various blade modes, the nonlinear equations of motion for the elastic bending and torsion of wind turbine blades are constructed. First mode in edgewise, two modes in flapwise and first mode in torsion directions are considered because these are the most prominent modes for wind turbine blades. Multiple-scales perturbation method is used to solve these equations. The analytical results of present study are compared with numerical results as well as those in the technical literature. The analytical model put forward in this paper may serve as a foundation for examining blade vibration and aeroelastic stability for various wind turbine designs.

II. Modal Problem Of The Coupled Vibration

Mode-shapes corresponding to the cantilever Euler-Bernoulli beam are considered. Therefore, the effects of rotary inertia, shear deformation and warping are neglected. It is assumed that the blade cross-sections remain planar. Since there is no warping of cross-section, any change of shape of the blade is due to rigid body motion. The *xyz* is the inertial coordinate system, attached to the root of the beam. The second coordinate system, $x_1 y_1 z_1$, defines the local coordinate system that origins at the barycenter Q of an arbitrary cross-section of the blade as depicted in Figure 1. Blade is in undeformed position when *x*- and x_1- axes are aligned. Some deformation will take place when *xyz* and $x_1 y_1 z_1$ coordinate systems are no longer aligned. Transformation that the blade will experience to get to the deformed state can be described in three counterclockwise rigid body rotations by Euler angles. Now take the variation of kinetic and potential energies in flapwise, edgewise and torsion directions using Hamilton's principle, the governing nonlinear equations for the forced vibration of blade can be derived as:

$$\rho A(x)\ddot{w} - \{F_T w'\}' - \rho A(x)\Omega^2 w \sin^2(\beta_p) + \rho A(x)\Omega^2[e_x + x\cos(\beta_p)]\sin(\beta_p) + 2\rho A(x)\Omega\sin(\beta_p)\dot{v} + \left\{\left(EI_{yy}(x) - EI_{zz}(x)\right) v'' \sin(\theta+\varphi)\cos(\theta+\varphi)\right\}'' + \left\{\left[EI_{yy}(x)\sin^2(\theta+\varphi) + EI_{zz}(x)\cos^2(\theta+\varphi)\right]w''\right\}'' = F_z - M'_y \quad (1)$$

$$\rho A(x)\ddot{v} - \{F_T v'\}' - \rho A(x)\Omega^2\left[e_y + v\right] - 2\rho A(x)\Omega\sin(\beta_p)\dot{w} + \left\{\left(EI_{yy}(x) - EI_{zz}(x)\right) w'' \sin(\theta+\varphi)\cos(\theta+\varphi)\right\}'' + \left\{\left[EI_{yy}(x)\cos^2(\theta+\varphi) + EI_{zz}(x)\sin^2(\theta+\varphi)\right]v''\right\}'' = F_y - M'_z \quad (2)$$

$$\rho A(x)\ddot{\varphi} - \{GJ\varphi'\}' - \left\{EA(x)k_A^2(\theta+\varphi)'\left[\frac{v'^2}{2} + \frac{w'^2}{2}\right]\right\}' + \rho A(x)e_I[\Omega^2 x(w'\cos\theta - v'\sin\theta)] + \left(EI_{yy}(x) - EI_{zz}(x)\right)\left[(w''^2 - v''^2)\sin\theta\cos\theta + v''w''\cos 2\theta\right] = M_x - \rho A(x)e_I\Omega^2\beta_p x\cos\theta \quad (3)$$

where ρ is the blade density, A is the cross-section area of blade segment, v is the edgewise deflection, w is the flapwise deflection, φ is the torsion deflection, EI_{yy} is the edgewise stiffness, EI_{zz} is the flapwise stiffness, GJ is the torsion stiffness, , F_T is the tensile force on the blade in the axial direction, θ is the twist angle, β_p is the conning angle, Ω is the blade rotational speed, and e_y is the directional component of bias e, F_y the edgewise aerodynamic force, F_z the flapwise aerodynamic force, M_x, M_y, M_z are the aerodynamic moments about *x*-axis, *y*-axis and *z*-axis, respectively as shown in Figure 2. The structural model needs to include at least the first edge mode, first two flap modes and the first torsion mode for a reliable dynamic analysis of wind turbine blade [11]. The modal expansion of equations is considered as:

$$v(x,t) = \gamma_1(x)p_1(t) \quad (4)$$

$$w(x,t) = \gamma_1(x)q_1(t) + \gamma_2(x)q_2(t) \quad (5)$$

$$\varphi(x,t) = \gamma_1(x)s_1(t) \quad (6)$$

where $p(t)$, $q(t)$ and $s(t)$ are the generalized coordinates in edge, flap and torsion directions, respectively. $\gamma_i(x)$ are the orthogonal modal functions. Mode shapes of a Euler-Bernoulli cantilever beam having fixed free boundary conditions are [12]:

$$\gamma_n(x) = cos\left(\beta_i \frac{x}{L}\right) - cosh\left(\beta_i \frac{x}{L}\right) - \left[\frac{cos\,\beta_i + cosh\,\beta_i}{sin\,\beta_i + sinh\,\beta_i}\right]\left[sin\left(\beta_i \frac{x}{L}\right) - sinh\left(\beta_i \frac{x}{L}\right)\right], i = 1,2 \quad (7)$$

where $\beta_1 = 1.8751$, $\beta_2 = 4.69409$. The final equations are obtained by reducing (1) - (3) through Galerkin's method as:

$$\ddot{q}_j = b_{j,1}\dot{q}_1 + b_{j,2}\dot{q}_2 + b_{j,3}q_1 + b_{j,4}q_2 + b_{j,5}q_1^2 q_2 + b_{j,6}q_1 q_2^2 + b_{j,7}q_1 p_1^2 + b_{j,8}q_2 p_1^2 + b_{j,9}q_1^3 + b_{j,10}q_2^3 + b_{j,11}\cos(\Omega t) + b_{j,12}q_1 s_1^2 + b_{j,13}q_2 s_1^2, \quad j = 1,2 \quad (8)$$

$$\ddot{p}_1 = g_1\dot{p}_1 + g_3 p_1 + g_4 p_1 q_1 q_2 + g_5 p_1 q_1^2 + g_6 p_1 q_2^2 + g_7 p_1^3 + g_8\cos(\Omega t) + g_9\,\mathrm{sins}(\Omega t) + g_{10} p_1 s_1^2 \quad (9)$$

$$\ddot{s}_1 = n_1\dot{s}_1 + n_3 s_1 + n_4 s_1 q_1 q_2 + n_5 s_1 p_1^2 + n_6 s_1 q_1^2 + n_7 s_1 q_2^2 + n_8\cos(\Omega t) \quad (10)$$

Coefficients of (8) - (10) are calculated using the structural values of NREL 5-MW reference wind turbine blade as defined in [13].

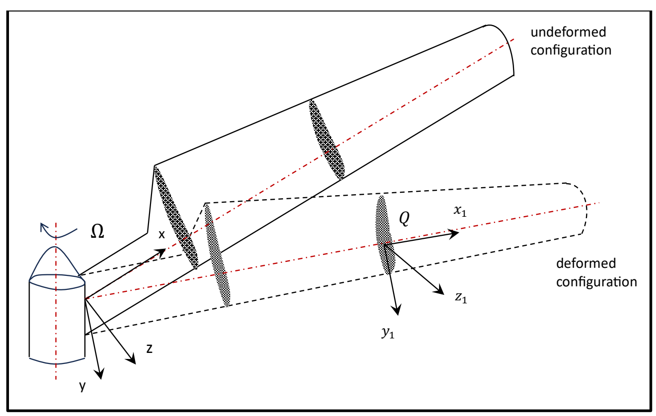


Figure 1. Undeformed and deformed configurations of wind turbine blade.

III. Multiple-scales Perturbation Solutions

Multiple-scales perturbation method is employed to solve (8) - (10). The aerodynamic loading is assumed to be the order of ε. Aerodynamic damping is assumed to be small so that it appears in the $O(\varepsilon^3)$ order terms:

$$q_j = \varepsilon q_{j,1} + \varepsilon^3 q_{j,3} + \cdots, \quad j = 1,2 \quad (11)$$

$$p_1 = \varepsilon p_{11} + \varepsilon^3 p_{13} + \cdots \quad (12)$$

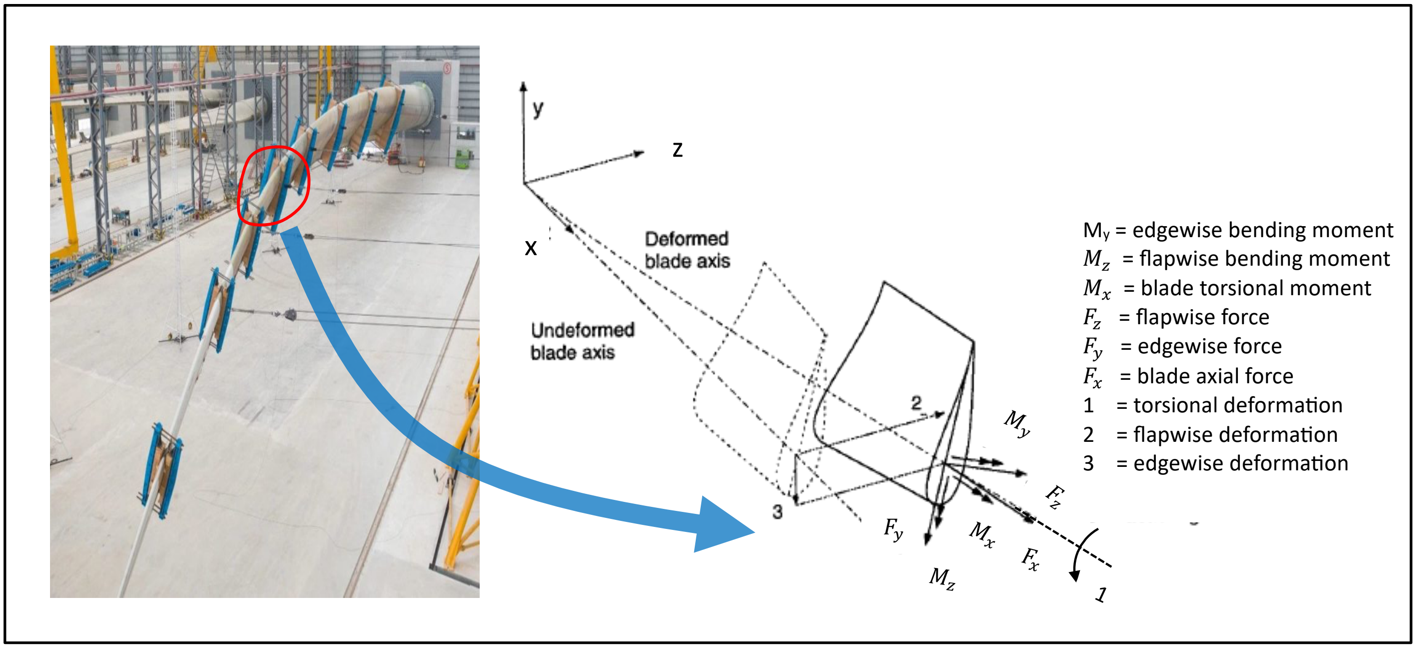


Figure 2. Forces acting on a blade segment and chordwise coordinate system.

$$s_1 = \varepsilon s_{11} + \varepsilon^3 s_{13} + \cdots \quad (13)$$

where ε is a small scaling parameter. Employing multiple-scales method on (8) - (10), one obtains:

$O(\varepsilon)$:

$$D_0^2 q_{j,1} - b_{j,3} q_{j,1} - b_{j,4} q_{2,1} = \cos(\Omega t), \quad j = 1,2 \quad (14)$$

$$D_0^2 p_{11} - g_3 p_{11} = g_8 \cos(\Omega t) + g_9 \sin(\Omega t) \quad (15)$$

$$D_0^2 s_{11} - n_3 s_{11} = n_8 \cos(\Omega t) \quad (16)$$

$O(\varepsilon^3)$:

$$D_0^2 q_{j,3} + 2D_0 D_2 q_{j,1} - b_{j,1} D_0 q_{1,1} - b_{j,2} D_0 q_{2,1} - b_{j,3} q_{1,3} - b_{j,4} q_{2,3} - b_{j,5} q_{1,1}^2 q_{2,1} - b_{j,6} q_{1,1} q_{2,1}^2 - b_{j,7} q_{1,1} p_{11}^2 - b_{j,8} q_{2,1} p_{11}^2 - b_{j,9} q_{1,1}^3 - b_{j,10} q_{2,1}^3 - b_{j,13} q_{1,1} s_{1,1}^2 - b_{14} q_{2,1} s_{11}^2 = 0, \quad j = 1,2 \quad (17)$$

$$D_0^2 p_{13} + 2D_0 D_2 p_{11} - g_1 D_0 p_{11} - g_3 p_{13} - g_4 p_{11} q_{1,1} q_{2,1} - g_5 p_{11} q_{1,1}^2 - g_6 p_{11} q_{2,1}^2 - g_7 p_{11}^3 - g_{11} p_{11} s_{11}^2 = 0 \quad (18)$$

$$D_0^2 s_{13} + 2D_0 D_2 s_{11} - n_1 D_0 s_{11} - m_3 s_{13} - m_4 s_{11} q_{1,1} q_{2,1} - m_5 s_{11} p_{11}^2 - m_6 s_{11} q_{11}^2 - m_7 s_{11} q_{2,1}^2 = 0 \quad (19)$$

Equations (14) - (19) leads to the first order approximation of the response as:

$$q_1 = \varepsilon a_m \cos(\omega_m t + \theta_m) + O(\varepsilon^3), \ m = 1,2 \quad (20)$$

$$q_2 = \varepsilon \Gamma_m a_m \cos(\omega_m t + \theta_m) + O(\varepsilon^3), \ m = 1,2 \quad (21)$$

$$p_1 = \varepsilon a_3 \cos(\omega_3 t + \theta_3) + O(\varepsilon^3) \quad (22)$$

$$s_1 = \varepsilon a_4 \cos(\omega_4 t + \theta_4) + O(\varepsilon^3) \quad (23)$$

where a_m and θ_m are the time dependent amplitudes and phase angles, ω_m are the natural frequencies of blade modes. Natural frequencies ω_m and the coefficients Γ_m are given below as:

$$\omega_m = \frac{1}{\sqrt{2}}\left[\sqrt{-(b_3 + c_4) \mp \sqrt{(b_3 + c_4)^2 - 4(b_3 c_4 - b_4 c_3)}}\right], \quad m = 1,2 \quad (24)$$

$$\omega_3 = \sqrt{-h_3}, \omega_4 = \sqrt{-m_3} \quad (25)$$

$$\Gamma_1 = \frac{1}{2(\Lambda_2 - \Lambda_1)}\left[\frac{\omega_1^2}{\omega_1^2 - \Omega^2}(b_{2,9} - b_{1,9}\Lambda_2) + \frac{\omega_2^2}{\omega_2^2 - \Omega^2}(b_{1,9}\Lambda_1 - b_{2,9})\right] \quad (26)$$

$$\Gamma_2 = \frac{1}{2(\Lambda_2 - \Lambda_1)}\left[\frac{\omega_1^2}{\omega_1^2 - \Omega^2}(b_{2,9} - b_{1,9}\Lambda_2)\Lambda_1 + \frac{\omega_2^2}{\omega_2^2 - \Omega^2}(b_{1,9}\Lambda_1 - b_{2,9})\Lambda_2\right] \quad (27)$$

$$\Lambda_k = -\left(\frac{b_{1,3} + \omega_k^2}{b_{1,4}}\right) = -\left(\frac{b_{2,3}}{b_{2,4} + \omega_k^2}\right), \ k = 1,2 \quad (28)$$

IV. Results And Discussions

Wind turbine blade model of NREL 5 MW is used to validate the natural frequencies associated with flapwise, edgewise and torsion deflections of the blade. Table 1 shows a comparison of

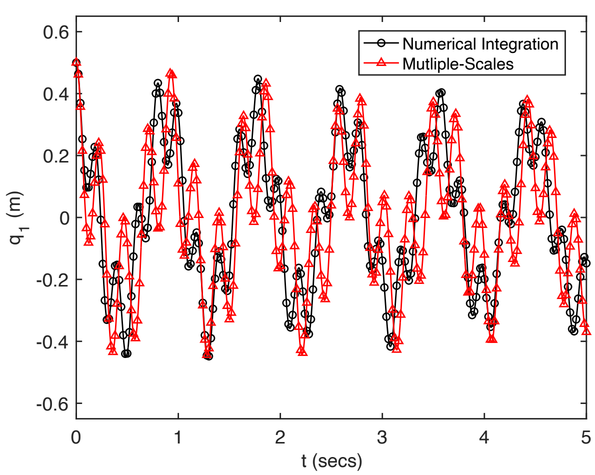
TABLE 1. NATURAL FREQUENCIES OF NREL 5MW WIND TRUBINE BLADE

Mode description	Frequency (HZ)					
	Jonkman et al. [13]	**Hansen [14]**	**Pourazarm et al. [15]**	**Meng [16]**	**Zhuang and Yuan [17]**	**Present Study**
First flapwise	0.6993	0.69	0.64	0.72	0.67	0.694
First edgewise	1.0898	1.08	-	1.07	-	1.13
Second flapwise	2.0205	1.8	1.86	2.05	1.96	1.89
Second edgewise	-	2.9	-	-	-	3.054
First torsion	-	8	5.39	5.62	5.71	5.5

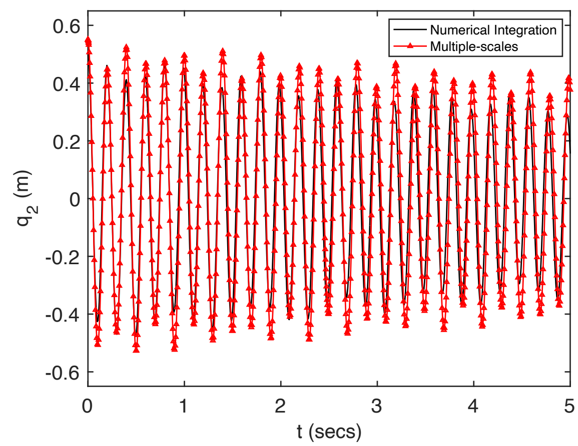
of the natural frequencies of NREL 5MW wind turbine blade as determined in this research alongside findings from other studies. The natural frequencies show a strong correlation with those reported in prior investigations.

Numerical and analytical (multiple-scales) results of non-resonant responses are presented in **Figure 3**. The analytical findings are subsequently compared with numerical results, which are achieved through the direct integration of (8) - (10) by employing the fourth-order Runge-Kutta technique with adjustable time steps. A significant correlation is observed between the outcomes. The discrepancies observed can largely be attributed to the omission of higher-order terms in the multiple-scale analysis. Overall, the current dynamical modeling and multiple-scales approach are effective for addressing both transient and steady-state responses of the blade under various resonance conditions.

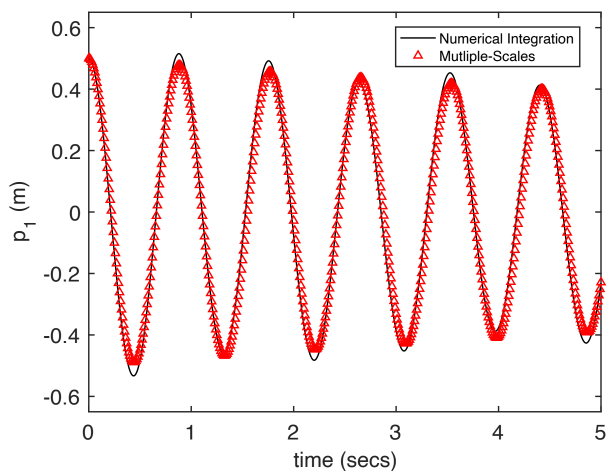
Further, the static aeroelastic response of a wind turbine blade is demonstrated in **Figure 4** along with comparisons to other studies. The steady-state deflection is computed at a rated wind speed with constant inflow and the absence of gravitational effects. The flapwise deflection closely aligns with the findings of **Jeong et al. (2009)**.



(a) First flapwise amplitude

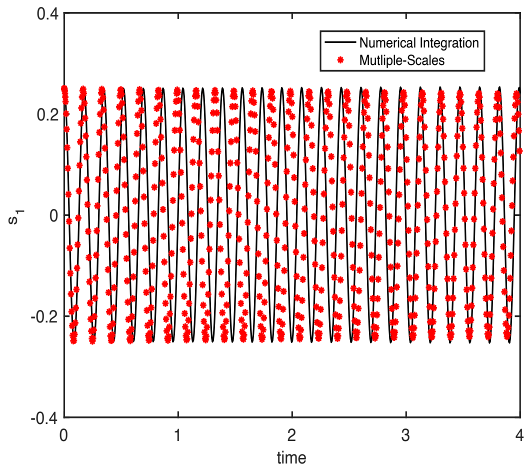


(b) Second flapwise amplitude



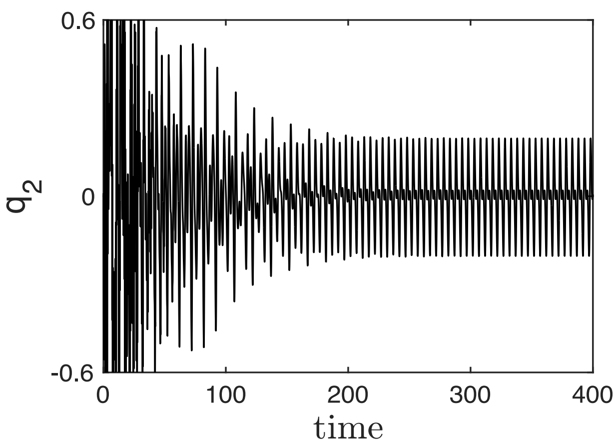
(c) First edgewise amplitude

The dynamics are studied in more detail in Figures 4 and 5 which display the phase diagrams and time history responses of NREL-5MW reference wind turbine blade at rotational speeds of 12.1 rpm and 16 rpm. It was noted that at rotor rated speed of $\Omega = 12.1$ rpm second flapwise mode is stable, flutter instability occurs at rotor speed of $\Omega = 16$ rpm.

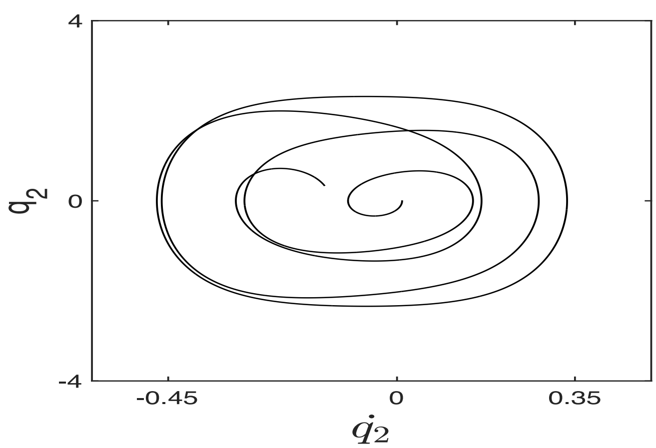


(d) First torsion amplitude

Figure 3. Verification of analytical responses through numerical results for amplitudes of vibration of various blade modes.

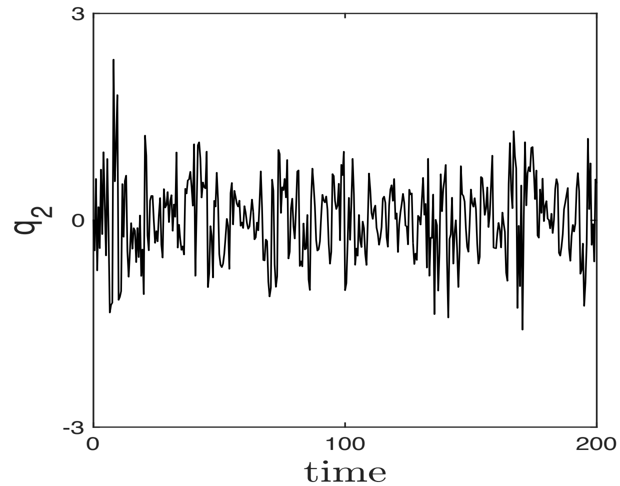


(a) Second flapwise amplitude (trajectories)

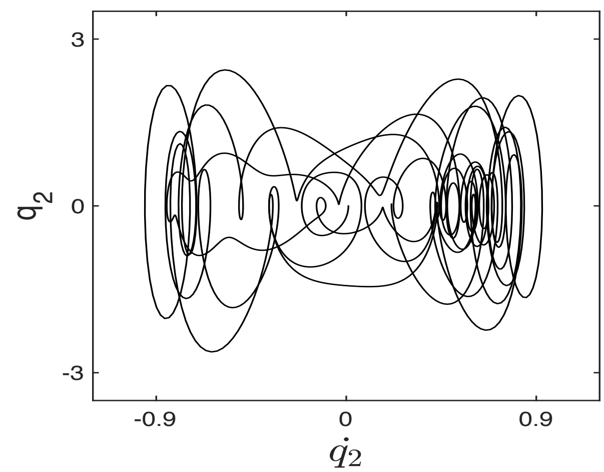


(v) Second flapwise amplitude (phase plot)

Figure 4. Time history graph and phase plot of modal amplitude for rotor speed $\Omega = 12.1\ rpm$.



(a) Second flapwise amplitude (trajectories)



(b) Second flapwise amplitude (phase plot)

Figure 5. Time history graph and phase plot of modal amplitude for rotor speed $\Omega = 16\ rpm$.

IV. Conclusion

Nonlinear vibration of a wind blade subjected to aerodynamic loading was examined, considering the coupling between first two flapwise, first edgewise and first torsion modes. Four coupled nonlinear equations were derived; two in flapwise, one in edgewise and one in torsion directions. The geometric nonlinearity and aerodynamic loading are included in the equations for a reference NREL 5-MW wind turbine blade. The analytical results of natural frequencies and amplitudes of motion from present study are in good agreement with results from technical literature. It was observed that multiple equilibria coexist, a limit cycle at low rotational speed of blade, while a limit cycle and an irregular motion at large blade speed. Second flapwise mode showed flutter instability at rotor speed of $\Omega = 16$ rpm.

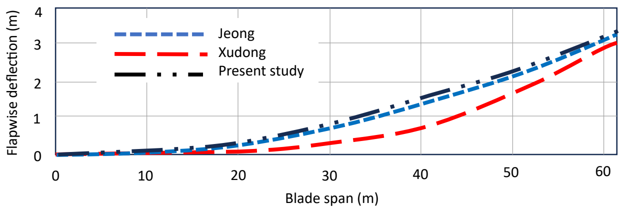


Figure 6. Comparison of flapwise amplitude with other studies.

References

[1] J. M. Park, H. Y. Park, S. Y. Jeong, S. I. Lee, Y. H. Shin and J. P. Park, “Linear vibration analysis of rotating wind-turbine blade,” Current Applied Physics, vol. 10, issue 2, pp. 332-334, 2010.

[2] J. Wang, D. Qin and Q. Zhang, “Mathematical model for predicting the blade behaviour of horizontal axis wind turbine,” Proceeding of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science, vol. 222 issue 9, pp. 1681-1694, 2008.

[3] M. H. Hsu, “Dynamic behavior of wind turbine blades,” Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science, vol. 222, issue 8, pp. 1453-1464, 2008.

[4] K. Stol, M. Balas and G. Bir, “Floquet modal analysis of a teetered-rotor wind turbine,” Journal of Solar Energy Engineering-Transactions of ASME, vol. 124, issue 4, pp. 364-371, 2002.

[5] A. Gangele and S. Ahmed, “Modal analysis of S809 wind turbine blade considering different geometrical and material parameters,” Journal of the Institution of Engineers (India): Series C, vol. 94, issue 3, pp. 225-228, 2013.

[6] A. Bercin and M. Tanaka, “Coupled flexural-torsional vibrations of Timoshenko beams,” Journal of Sound and Vibration, vol. 207, issue 1, pp. 47-59, 1997.

[7] R. Bishop and W. G. Price, “Coupled bending and twisting of a Timoshenko beam,” Journal of Sound and Vibration, vol. 50, issue 4, pp. 469-477, 1977.

[8] K. B. Subrahmanyam, S. V. Kulkarni and J. S. Rao, “Coupled bending-torsion vibrations of rotating blades of asymmetric aerofoil cross section with allowance for shear deflection and rotary inertia by use of the Reisner method,” Journal of Sound and Vibration, vol. 75, issue 1, pp. 17-36, 1981.

[9] A. Kumar, A. Dwivedi, V. Paliwal and P. P. Patil, “Free vibration analysis of Al 2024 wind turbine blade designed for Uttarakhand region based on FEA,” Procedia Technology, vol. 14, pp. 336-347, 2014.

[10] M. U. Saram and J. Yang, “A study on coupled edgewise and flapwise vibration modes of wind turbine blade,” Journal of Vibration and Control, vol. 0, 2024.

[11] J. G. Holierhoek, “Advances in wind turbine blade design and materials,” 1st Edition, Elsevier Publishers, 2013.

[12] P. Hagedorn and A. Dasgupta, "Vibrations and waves in continuous mechanical systems," Southern Gate, Chichester: John Wiley & Sons Ltd, 2007.

[13] J. Jonkman, S. Butterfield, W. Musial and G. Scott, "Definition of a 5-MW reference wind turbine for offshore system development," National Renewable Energy Laboratory, 2009.

[14] M. H. Hansen, “Stability analysis of three-bladed turbines using an eigenvalue approach,” Aerospace Sciences Meeting and Exhibit, Proceedings of the ASME Wind Energy Symposium, Reno, Nevada, 2004.

[15] P. Pourazarm, Y. M. Sadeghi and M. A. Lackner, “A parametric study of coupled-mode flutter for MW-size wind turbine blades,” Wind Energy, vol. 19, issue 3, pp. 497–514, 2016.

[16] F. Meng, “Aeroelastic stability analysis for large-scale wind turbines,” PhD Thesis Dissertation, TU Delft, 20111.

[17] Y. Zhuang and G. Yuan, “Study on coupled mode flutter parameters of large wind turbine blades,” Scientific Reports, 2024.

[18] M. S. Jeong, I. Lee, S. J. Yoo and K. C. Park, “Torsional stiffness effects on the dynamics stability of a horizontal axis wind turbine blade,” Energies, vol. 6, issue 4, pp. 2242–61, 2013.

[19] W. Xudong, W. Z. Shen, W. J. Zhu, J. N. Sorensen and C. Jin, “Shape optimization of wind turbine blades,” Wind Energy, vol. 12, issue 8, pp. 781–803, 2009.