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Abstract: A production system consisting of two parallel machines with production-dependent failure rates is investigated in this paper. The machines produce one type of final product and unmet demand is backlogged. The objective of the system is to find a productivity policy for both machines that will minimize the inventory and shortage costs over an infinite horizon. The failure rate of the main machine depends on its productivity, while the failure rate of the second machine is constant. In the proposed model, the main machine is characterized by a higher productivity. This paper proposes a stochastic dynamic programming formulation of the problem and derives the optimal policies numerically. A numerical example is included and sensitivity analyses with respect to the system parameters are examined to illustrate the importance and effectiveness of the proposed methodology.

Keywords: Production planning; Stochastic dynamic programming; Numerical methods.

1. Introduction

The number of scientific publications covering failure-prone manufacturing systems has been growing steadily as a result of the intensive search for increased productivity and better customer service. A complete analytical solution was given in Akella and Kumar (1986), for a manufacturing system characterized by a homogeneous Markov process. The authors showed that the hedging point policy is the optimal control policy for minimizing discounted cost. In such a policy, the machine operates at a maximal rate until the inventory hits a safety stock level. If the current inventory level exceeds this level, no production should be carried out, but if it is equal to this level, then production should be just enough to meet demand. For a single machine, single part-type system, the expression of the optimal safety stock level was derived by Akella and Kumar (1986). This basic result has been extended in several ways over the years, with most such extensions relating to the Markovian case (see Tan and Gershwin (2004), Dong-Ping (2009), etc.). Only a few papers have examined semi-Markov processes (see Hu and Xiang (1995); Dehayem et al. (2011); Kazaz and Sloan (2013) etc.). In the Markovian case, however, a frequent assumption is that the underlying Markov process is homogeneous. The assumption in the semi-Markov processes is that the system deteriorates with age and number of failures. While these are reasonable assumptions, which in some cases provide simple and appealing mathematical solutions, the authors did not address the question of what happens if the machine is used to its maximum production capacity for a long period. The problem becomes much more pertinent if the failure rate depends on the productivity. In Rishel (1991), it was proven that the hedging point policy remains optimal if and only if the dependence of the failure rate on productivity is quadratic.

Similarly, one of the most important achievements of the research of Hu et al. (1994) was the investigation of the necessary and sufficient conditions for the optimality of the hedging point policy for a single machine, single part-type problem, when the failure rate of the machine is a function of productivity. They showed that the hedging point policies are only optimal under linear failure rate functions. As per their discussion, numerical results in the general case suggest that as the inventory level approaches a hedging level, it may be beneficial to decrease productivity in order to realize gains in reliability. This conjecture was confirmed by the numerical results reported in Martinelli (2007), where the author considered a long average cost function and a machine characterized by two failure rates: one for low and one for high productivities. Martinelli (2010) generalizes the problem of Martinelli (2007) by considering one machine with different failure rates: more specifically, the failure rate is assumed to depend on productivity, through an increasing, piecewise constant function. Dahane et al (2012) studied the problem of dependence between production and failure rates in the context of a multi-product manufacturing system. The system is analysed in discrete time and the results provided an answer about how to produce and what to produce over a finite horizon. The authors considered a manufacturing system consisting of a single randomly failing and repairable machine producing two products. A method for integrating load distribution decisions and production planning in the context of multi-state systems was presented by Nourelfath and Yalaoui (2012). The authors considered the load versus failure rate relationship while optimizing planning of production systems. Their integrated objective was to minimize the sum of capacity change costs, unused capacity costs, setup costs, holding costs, backorder costs, and production costs over a finite horizon.

A stochastic deteriorating production system consisting of two parallel machines with the productivity-dependent failure rates of the main machine is investigated in this paper. The stochastic nature of the system is due to machines that are subject to a non-homogeneous Markov process resulting from the dependence of failure rates on the production rate (productivity). The machines produce a single part type. Whenever a breakdown occurs, a corrective maintenance is performed. A repair action renews the machines. Our objective is to find the productivities of both machines such as to minimize the inventory and the shortage costs over an infinite horizon. To solve the optimization problem, we propose a stochastic dynamic programming formulation and derive the optimal production policies numerically. Numerical examples are included and sensitivity analyses with respect to the system parameters are also examined to illustrate the significance and effectiveness of the proposed methodology. As an extension, we apply this methodology to discuss the optimal productivities of manufacturing systems consisting of two machines with five failure rates depending on the productivity of the main machine.

This work distinguishes itself from the literature in three ways. First, the paper extends the work of Liberopoulos and Caramanis (1994), Martinelli (2010) and Dahane et al (2012) to manufacturing systems consisting of more than one machine subject to a non-homogeneous Markov process with productivity-dependent failure rates. We also extend the work of Dahane et al. (2012) and, Nourelfath and Yalaoui (2012) to the production planning over an infinite horizon. Secondly, the case of manufacturing systems consisting of two machines with multiple failure rates is discussed. Lastly, we study the possible industrial applications of the formulation and the approaches used.

The rest of this paper is broken down as follows. In Section 2, we present the industrial context of the problem under study. Section 3 covers notations and assumptions used in this research, and presents the problem statement. In Section 4, numerical results and sensitivity analyses are presented. Section 5 examines an extension to the case of multiple failure rates. Discussions and policies implementation are presented in Section 6, and the paper is finally concluded in Section 7.

2. Industrial context

The formulation, the approaches, and the numerical procedures used in this paper could be applied to many industries in which machines can be subjected to random failures and their production rates can also be controlled. The phenomenon has been experienced in machinery and mechanical assemblies, including automobile, aircraft engine and machine tools, and paper manufacturing plants. For example, in the metallic parts machining industries, where basic turning lathes and computer numerically controlled (CNC) lathes are used, the reliability of the machine-tools will depend on how they are used – the type of workpieces, cutting tools, process parameters selected.

The most basic turning lathe is the engine lathe, which is used for single, prototype, and low-quantity parts. The major lathe used in production today is the CNC lathe. Such lathes can produce a variety of parts requiring surfacing, turning, boring, grooving, drilling, threading, and chamfering in single or combined motions.

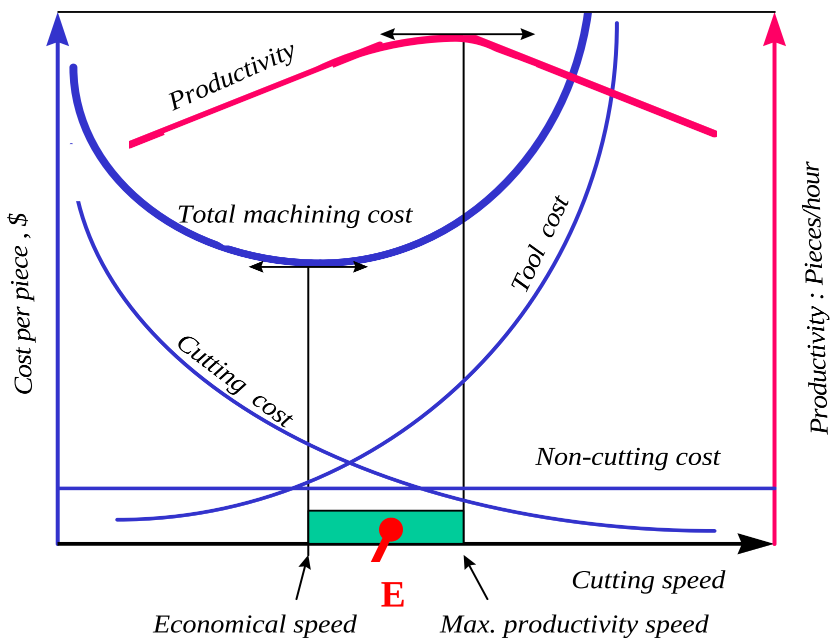


Fig. 1: Schematic representation of machining costs and productivity as a function of the cutting speed (adapted from Groover, 2007)

Any motion that can be expressed mathematically can be programmed into the lathe's computer control. CNC lathes machining provide parts characterized by great precision and low variability. It allows the machining of mechanical parts at high cutting speeds, which improves the productivity and the part surface finish. However, high speed machining (HSM) has some disadvantages: For instance higher acceleration and deceleration rates require precise forecasting and highly capable controllers. As well, constant spindle starting and stopping results in faster wear of guide ways, ball screws and spindle bearings, leading to higher maintenance costs. HSM also requires specific process knowledge, programming equipment and interfaces for the fast data transfer needed. Finding suitably trained staff can be difficult, and HSM can involve a considerable "trial and error" period. Good work and process planning is necessary, along with significant safety precautions and safety enclosing (bullet-proof covers). Tools, adapters and screws need to be checked regularly for fatigue cracks. Only tools with posted maximum spindle speeds can be used.

The dependency of the machining cost and productivity (parts per hour) as a function of the cutting speed are presented in Fig. 1. Examining Fig. 1, we see that to minimize production costs and take into account the reliability of the CNC lathe, it would be advantageous to reduce the machining speed from its maximal productivity value to its economical value (see zone E).The machining costs are broken down into:

- Non-cutting costs (loading, unloading, assembly, rapid movements of approach, return to the table). These costs are independent of the cutting parameters.
- Tool costs (purchasing price, tool holders, tool changing costs, resharpening costs). When machining at higher speeds, tools wear out quickly, leading to short tool life and frequent tool change.
- Cutting costs (real metal removal cost). These decrease when the speed increases.

The total machining costs, which are at their lowest at the speed called the *economical speed*, represent the sum of all tool costs, cutting costs and non-cutting costs. Similarly, the productivity varies with the cutting speed and is at a maximum at the speed called the *maximum productivity speed*. Two important observations should be made here. First, the cost increase is due mainly to tool costs related to machine maintenance and repairs resulting from tool deterioration and operator mistakes induced by insufficient training. Secondly, in the shaded area, the total machining cost is a growing function of productivity. At the level of production optimization, details such as the machine speed cannot be taken into account, but the described phenomena can be addressed by considering the machine failure rate as dependent on the machine's productivity. Below, we formulate an optimization model and develop appropriate techniques for its solution.

3. Problem statement and optimality conditions

Before delving into the problem statement, we first present the notations and assumptions used throughout this article.

3.1 Notations

The model under consideration is based on the following notations:

u_1: productivity of the main machine M_1
u_2: productivity of the second machine M_2
$u_{1\max}$: maximal productivity of M_1
U: economical productivity (in terms of machine's reliability) of M_1
$u_{2\max}$: maximal productivity of M_2
x: stock level
d: customer demand rate
ξ: stochastic process (manufacturing system)
c^+: inventory cost
c^-: backlog cost
$\lambda_{\alpha\beta}$: transition rate from mode α to mode β
Q: transition rate matrix
π: vector of limiting probabilities
$g(\cdot)$: instantaneous cost function
$J(\cdot)$: total cost

$\nu(\cdot)$: value function
ρ: discount rate
n: number of failure rates

3.2 Assumptions

This section presents the assumptions used throughout this paper.

(1) For the considered two-machine single-product environment, the machines are subject to random breakdowns and repairs. The failure rate of one machine depends on its productivity. This assumption represents the original characteristic of our approach. Other works consider one machine with a productivity-dependent failure rate or the system that deteriorates with age and number of failures.
(2) The shortage cost depends on parts produced for backlog (average value ($/unit)).
(3) The inventory cost depends on parts produced for positive inventory (average value ($/unit)). Assumptions 2 and 3 are common in inventory management.
(4) The productivity of the main machine is higher than that of the second machine.
(5) The second machine alone cannot satisfy customer demand. This machine is a supporting machine. The main machine is unable to satisfy customer demand with its economical productivity, which is why another machine (second machine) is called upon.

3.3 Problem formulation

As illustrated in Fig. 2, the manufacturing system studied consists of two parallel machines denoted as M_1 and M_2, which produce a single part type. When the main machine works at a faster rate, it is more likely to fail. The mode of the machine M_i can be described by a stochastic process $\xi_i(t),\ i=1,2$ with value in $B_i=\{1,2\}$. Such a machine is available when it is operational ($\xi_i(t)=1$) and unavailable when it is under repair ($\xi_i(t)=2$). The transition diagram, which describes the dynamics of the considered manufacturing system, is presented in Fig. 3. We then have $\xi(t)\in B=\{1,2,3,4\}$. With $\lambda_{\alpha\beta}$ denoting a jump rate of the system from state α to state β, we can describe $\xi(t)$ statistically by the following state probabilities:

$$\mathrm{P}\left[\xi(t+\delta t)=\beta \middle| \xi(t)=\alpha\right]=\begin{cases}\lambda_{\alpha\beta}(.)\,\delta t+o(\delta t) & if\,\alpha\neq\beta\\ 1+\lambda_{\alpha\beta}(.)\,\delta t+o(\delta t) & if\,\alpha=\beta\end{cases}$$

(1)

where $\lambda_{\alpha\beta}\geq 0\quad(\alpha\neq\beta)$, $\lambda_{\alpha\alpha}=-\sum_{\beta\neq\alpha}\lambda_{\alpha\beta}$ and $\lim_{\delta t\to 0}\frac{o(\delta t)}{\delta t}=0$ for all $\alpha,\beta\in B$.

The operational mode of the manufacturing system can be described by the random vector $\xi(t)=(\xi_1(t),\xi_2(t))$. Given that the dynamics of each machine is described by a 2-state stochastic process, the set of possible values of the process $\xi(t)$ can be determined from the values of $\xi_1(t)$ and $\xi_2(t)$ as illustrated in Table 1, with:

- Mode 1: M_1 and M_2 are operational
- Mode 2: M_1 is operational and M_2 is under repair
- Mode 3: M_1 is under repair and M_2 is operational
- Mode 4: M_1 and M_2 are under repair

Table 1. Modes of a two-machine manufacturing system

$\xi_1(t)$	1	1	2	2	Machine 1	Stochastic process
$\xi_2(t)$	1	2	1	2	Machine 2	Stochastic process
$\xi(t)$	1	2	3	4	Manufacturing system	Stochastic process

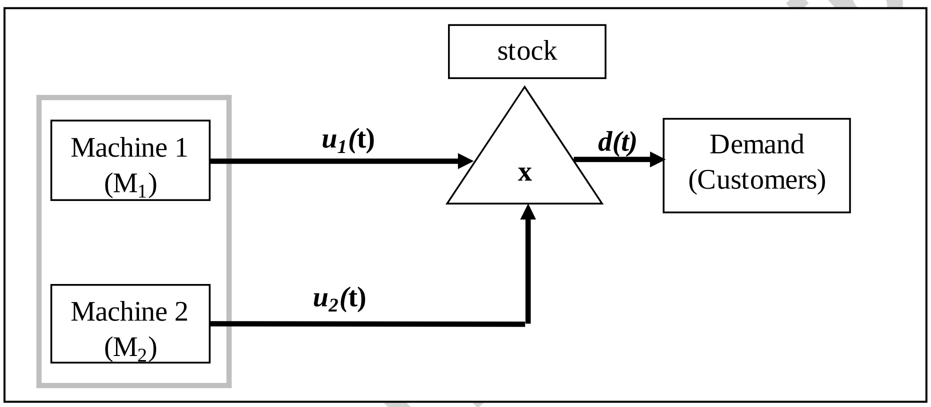


Fig. 2: Structure of the production system

The dynamics of the system is described by a discrete element, namely $\xi(t)$, and a continuous element $x(t)$. The discrete element represents the status of the machines and the continuous one represents that of the stock level. It can be positive for an inventory or negative for a backlog.

We assume that the failure rate of M_1 depends on its productivity, and is defined by:

$$q_{12}^1 = \begin{cases} \theta_1 & \text{if } u_1 \in (U, u_{1\max}] \\ \theta_2 & \text{if } u_1 \in [0, U] \end{cases} \quad \text{with } \theta_1 \geq \theta_2 \geq 0 \text{ and } 0 \leq U \leq u_{1\max}$$

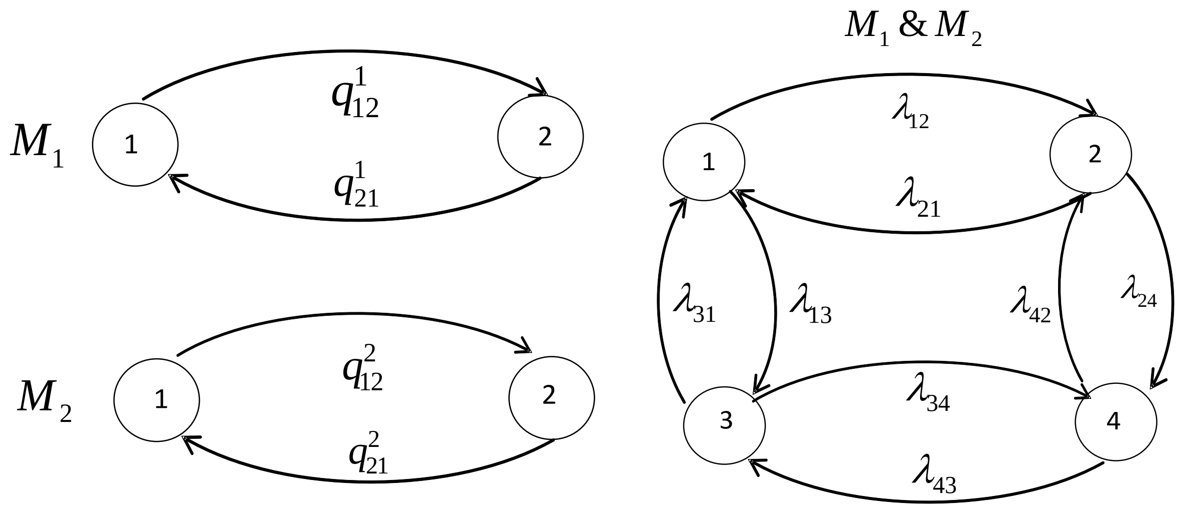
Hence, $\xi(t)$ is described by the following matrix:

$$Q = \begin{cases} \Theta_1 & \text{if } u_1 \in (U, u_{1\max}] \\ \Theta_2 & \text{if } u_1 \in [0, U] \end{cases} \quad \text{with}$$

(2)

$$\Theta_1 = \begin{pmatrix} -(q_{12}^2+\theta_1) & q_{12}^2 & \theta_1 & 0 \\ q_{21}^2 & -(q_{21}^2+\theta_1) & 0 & \theta_1 \\ q_{21}^1 & 0 & -(q_{21}^1+q_{12}^2) & q_{12}^2 \\ 0 & q_{21}^1 & q_{21}^2 & -(q_{21}^1+q_{21}^2) \end{pmatrix}; \text{ with } \theta_1 = \lambda_{13} = \lambda_{24}$$

$$\Theta_2 = \begin{pmatrix} -(q_{12}^2+\theta_2) & q_{12}^2 & \theta_2 & 0 \\ q_{21}^2 & -(q_{21}^2+\theta_2) & 0 & \theta_2 \\ q_{21}^1 & 0 & -(q_{21}^1+q_{12}^2) & q_{12}^2 \\ 0 & q_{21}^1 & q_{21}^2 & -(q_{21}^1+q_{21}^2) \end{pmatrix}; \text{ with } \theta_2 = \lambda_{13} = \lambda_{24}$$



a) States transition of each machine b) States transition of the system

$q_{12}^1 = \lambda_{13} = \lambda_{24}$ (failure rate of M_1)

$q_{12}^2 = \lambda_{12} = \lambda_{34}$ (failure rate of M_2)

$q_{21}^1 = \lambda_{31} = \lambda_{42}$ (corrective maintenance rate of M_1)

$q_{21}^2 = \lambda_{21} = \lambda_{43}$ (corrective maintenance rate of M_2)

Fig. 3: States transition diagram of the considered system

The continuous part of the system dynamics is described by the following differential equation:

$$\frac{dx(t)}{dt} = u_1(t) + u_2(t) - d, \; x(0) = x_0 \tag{3}$$

where x_0 and d are the given initial stock level and demand rate, respectively.

The set of the feasible control policies A, including $u_1(\cdot)$ and $u_2(\cdot)$, is given by:

$$\mathrm{A} = \left\{ \left(u_1(\cdot), u_2(\cdot) \right) \in \Re^2, 0 \le u_1(\cdot) \le u_{1\max}, 0 \le u_2(\cdot) \le u_{2\max} \right\} \tag{4}$$

where $u_1(\cdot)$ and $u_2(\cdot)$ are known as control variables, and constitute the control policies of the problem under study. The maximal productivities of the main machine and the second machine are denoted by $u_{1\max}$ and $u_{2\max}$, respectively.

Let $g(\cdot)$ be the cost rate defined as follows:

$$g(\alpha,x)=c^{+}x^{+}+c^{-}x^{-}$$
(5)

where constants c^{+} and c^{-} ($ per part per unit of time) are used to penalize inventory and backlog respectively, $x^{+}=\max(0,x), x^{-}=\max(-x,0)$.

The production planning problem considered in this paper involves the determination of the optimal control policies ($u_1^*(t)$ and $u_2^*(t)$) minimizing the expected discounted cost $J(\cdot)$ given by:

$$J(\alpha,x,u_1,u_2)=E\left\{\int_0^{\infty}e^{-\rho t}g(\alpha,x)dt\,\middle|\,x(0)=x_0,\xi(0)=\alpha\right\}$$
(6)

where ρ is the discount rate. The value function of such a problem is defined as follows:

$$v(\alpha,x)=\inf_{(u_1(\cdot),u_2(\cdot))\in\mathrm{A}(\alpha)}J(\alpha,x,u_1,u_2)\quad\forall\alpha\in B$$
(7)

The properties of the value function and the manner in which the Hamilton-Jacobi-Bellman (HJB) equations are obtained can be found in Kenné et al. (2003), with a constant failure rate.

3.4 Optimality conditions

Regarding the optimality principle, we can write the HJB equations as follows:

$$\rho v(\alpha,x)=\min_{(u_1,u_2)\in\mathrm{A}(\alpha)}\left[g(\alpha,x)+\sum_{\beta\in B}\lambda_{\alpha\beta}v(\beta,x)+(u_1+u_2-d)\frac{\partial v(\alpha,x)}{\partial x}\right]$$
(8)

where $\frac{\partial v(\alpha,x)}{\partial x}$ is the partial derivative of the value function $v(\alpha,x)$

The optimal control policy $\left(u_1^*(\cdot),u_2^*(\cdot)\right)$ denotes a minimizer over A of the right hand of Eq. (8). This policy corresponds to the value function described by Eq. (7). When the value function is available, an optimal control policy can then be obtained by solving Eq. (8). However, an analytical solution of Eq. (8) is almost impossible to obtain. The numerical resolution of the HJB Eq. (8) represents a challenge which was considered insurmountable in the past. Boukas and Haurie (1990) showed that implementing Kushner's method can solve such a problem in the context of production planning. In the Appendix, we present the numerical methods used to solve the proposed optimality conditions. In this research, the contribution to the optimality conditions lies in the fact that at modes 1 and 2, where M_1 is operational, we developed modified HJB equations with additional equations due to the consideration of multiple failure rates. Hence, for the examples of two, three, five and nine failure rates (as in the numerical examples presented in sections 4 and 5), we obtained multiple HJB equations with four, six, ten and eighteen equations instead of two, three, five and nine as in the case of a manufacturing system without a productivity-dependent failure rate. The reader is referred to equations (A.2) and (A.3) for the case of two failure rates. The next section provides a numerical example to illustrate the structure of the control policies.

4. Simulation and numerical example

Here, we illustrate the resolution of the model above with a numerical example. Sensitivity analyses with respect to the system parameters are also presented to illustrate the importance and effectiveness of the proposed methodology.

4.1 Numerical results

In this section, we present a numerical example for the manufacturing system presented in Section 3. A four-state Markov process with the modes in $B=\{1,2,3,4\}$ describes the system capacity. The instantaneous cost is described by Eq. (5).

The considered computation domain D is given by:

$$D=\{x:-20\leq x\leq 40\}$$

(9)

The limiting probabilities of modes 1, 2, 3 and 4 (i.e., π_1, π_2, π_3 and π_4) are computed as follows:

$$\pi\cdot Q(\cdot)=0 \quad \text{and} \quad \sum_{i=1}^{4}\pi_i=1$$

(10)

where $\pi=(\pi_1,\pi_2,\pi_3,\pi_4)$ and $Q(\cdot)$ is the corresponding 4×4 transition rate matrix given by Eq. (2).

Table 2. Parameters of numerical example

c^+	c^-	h	U	$u_{1\max}$	$u_{2\max}$	d	θ_1	θ_2	q_{12}^2	q_{21}^1	q_{21}^2	ρ
1	50	3	0.75	1.2	0.65	1	0.03	0.02	0.04	0.1	0.2	0.03

The condition for meeting customer demands, over an infinite horizon is given by:

$$\pi_1\cdot(U+u_{2\max})+\pi_2\cdot U+\pi_3\cdot u_{2\max}>d$$

(11)

where (π_1,π_2 and π_3) constitute the limiting probability at the operational modes of the machines. Eq. (11) is also satisfied with $u_{1\max}$ because $U<u_{1\max}$. Table 2 summarizes the parameters of the numerical example for which the feasibility conditions given by Eq. (11) are satisfied.

The productivities at mode 1 of machines M_1 and M_2 are presented in Figs. 4 and 5, respectively. Examining these figures, we can see that the threshold z_1 is low because both machines are operational. The results show that the productivities are set to zero for comfortable stock levels. At this point, there is no need to produce parts to ensure comfortable stock levels. According to the classical results as in Kenne et al. (2012) and references therein, the computational domain is expected to be divided into two stages, such as in Fig. 5. Our results show however that the computational domain of Fig. 4 is divided into three stages, which represents a specific finding of this paper. The optimal production control policy consists of one of the following rules for M_1:

1. Set the productivity of M_1 to its maximal value when the current stock level is under the first threshold value ($z_1 = 4.0$);
2. Reduce the productivity of M_1 to its economical value when the current stock level approaches the second threshold value ($z_2 = 19.0$);
3. Set the productivity of M_1 to zero when the current stock level is greater than the second threshold value.

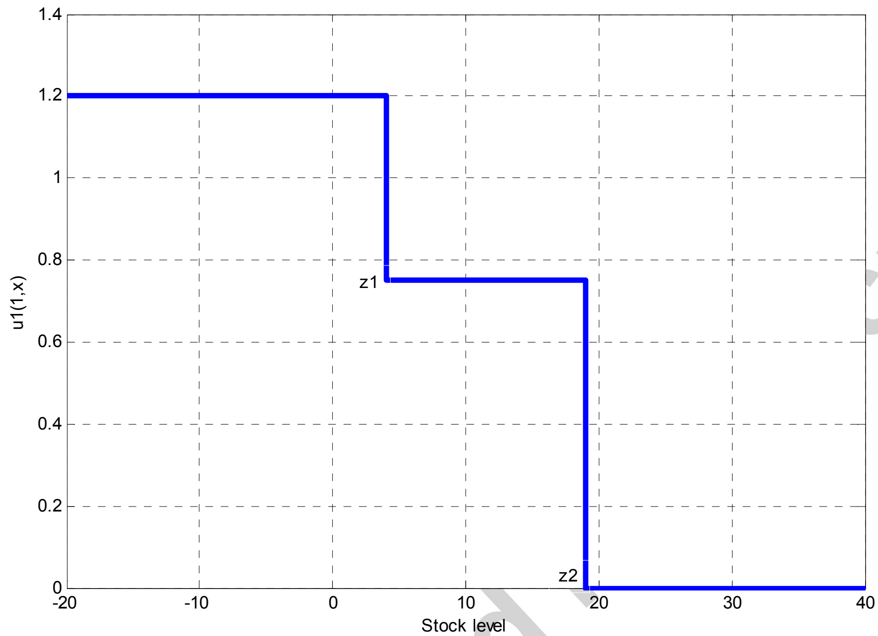


Fig. 4: Productivity of M_1 at mode 1

The control policies obtained in Fig. 4 are of multi-hedging point policy form. According to these results, the optimal productivities for the two machines can be expressed as follows:

$$u_1(x,1) = \begin{cases} u_{1\max} & \text{if } x < z_1 \\ U & \text{if } z_1 \le x < z_2 \\ 0 & \text{if } x > z_2 \end{cases}$$

(12)

where z_1 and z_2 are the first and the second threshold values of M_1, respectively.

$$u_2(x,1) = \begin{cases} u_{2\max} & \text{if } x < z_2 \\ 0 & \text{if } x > z_2 \end{cases}$$

(13)

where z_2 is the optimal threshold value at mode 1.

The productivity of M_1 at mode 2 of the system is presented in Fig. 6. Unlike the case illustrated in Fig. 4, where the tendency was to use the maximal productivity of the main machine less, at mode 2, the first threshold ($z_3 = 13.0$) is higher than z_1 in Fig. 4 because the machine works alone. However, the control policy is still a multi-hedging point policy, and is defined by:

$$u_1(x,2) = \begin{cases} u_{1\max} & \text{if } x < z_3 \\ U & \text{if } z_3 \le x < z_4 \\ 0 & \text{if } x > z_4 \end{cases}$$

(14)

where z_3 and z_4 are the first and second threshold values of M_1 at mode 2, respectively.

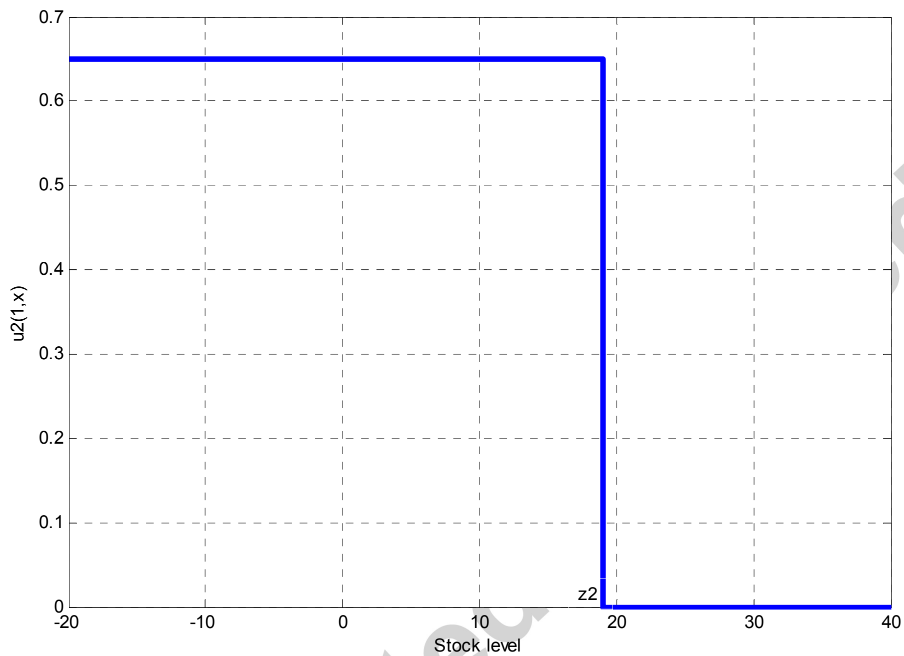


Fig. 5: Productivity of M_2 at mode 1

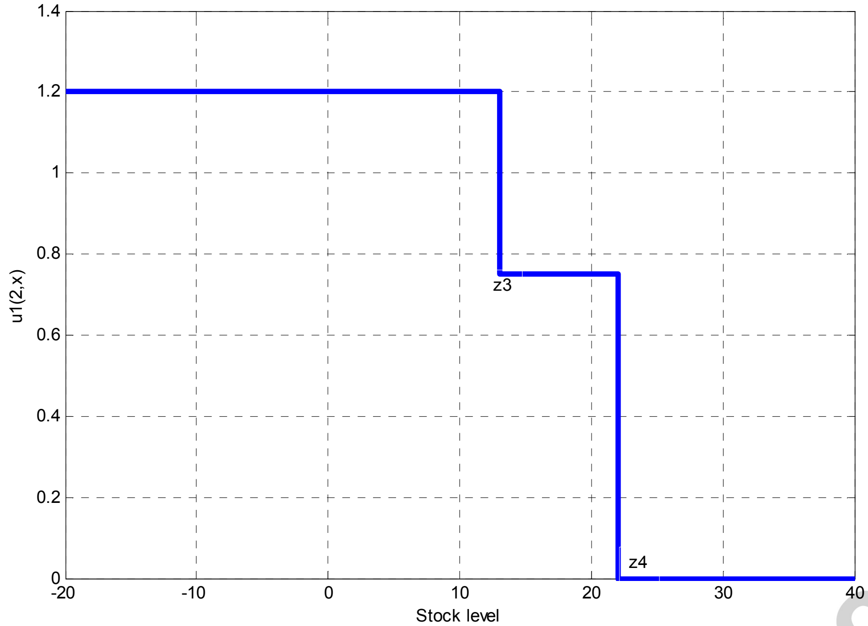


Fig. 6: Productivity of M_1 at mode 2

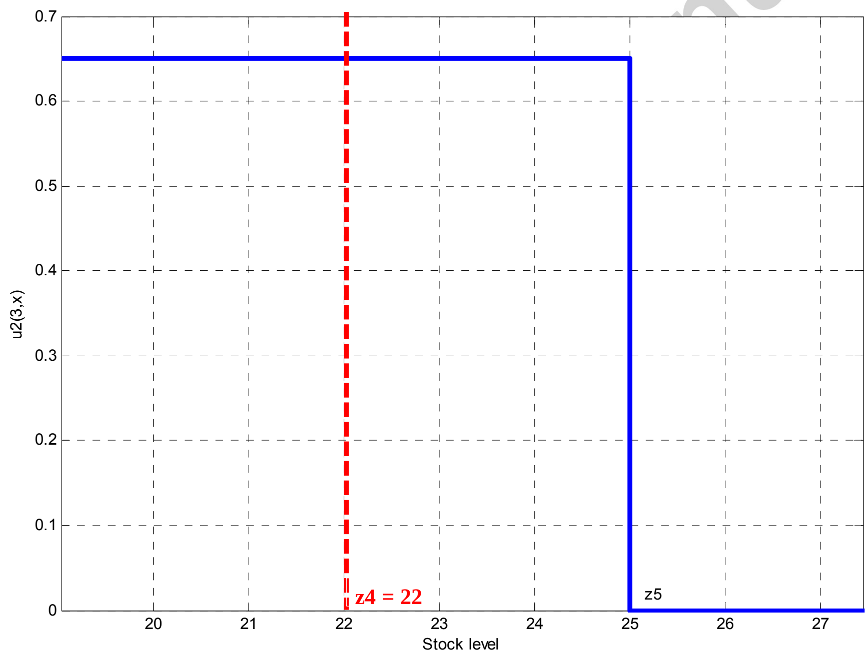


Fig. 7: Productivity of M_2 at mode 3

The productivity of M_2 at mode 3 is plotted in Fig. 7. The results of this figure show that the threshold value ($z_5 = 25$) is higher than the thresholds z_2 and z_4 because at mode 3, M_1 is under repair. The second machine must use its maximum productivity over a long period to avoid over-shortages. With numerical methods, the results show $z_5 = 25$. However in the industrial context, the system cannot exceed the value of $z_4 = 22$. Hence, the threshold value z_5 will be ignored.

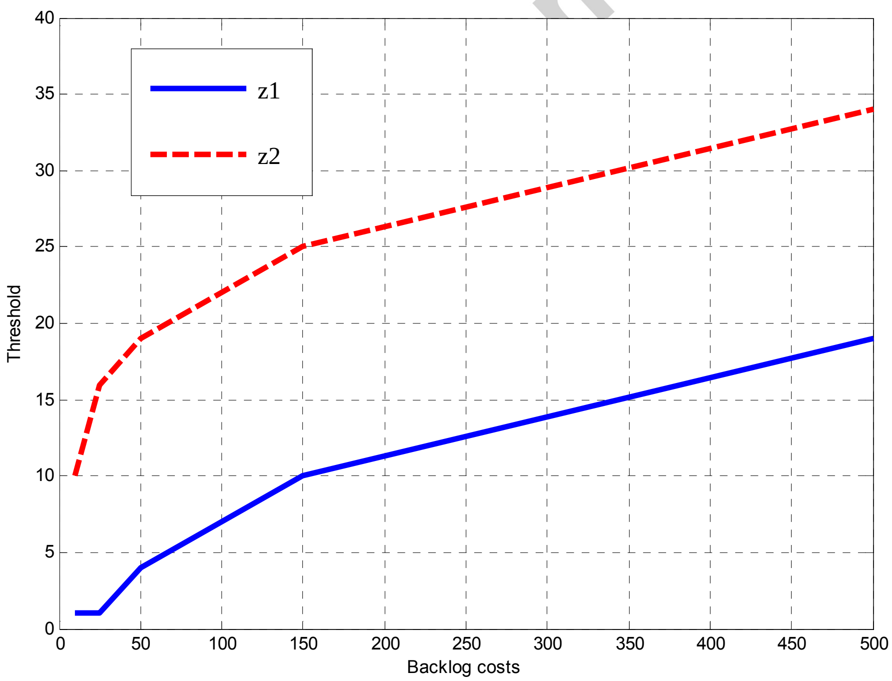
In the manufacturing system consisting of two machines and one type of product, with a constant failure rate, the optimal control policy is characterized by two threshold values (Ouaret et al., 2013). The results obtained in this paper show that the optimal control policy is characterized by four different threshold parameters (z_1,z_2,z_3 and z_4) due to the fact that the main machine degrades according to its productivity speed. This is a main finding of this paper.

The next section analyzes the sensitivity of the policies obtained with respect to the various parameters of the model. Several experiments were conducted to ensure that the structure of the obtained policies is maintained under parameter variation, and therefore, can be used in practice.

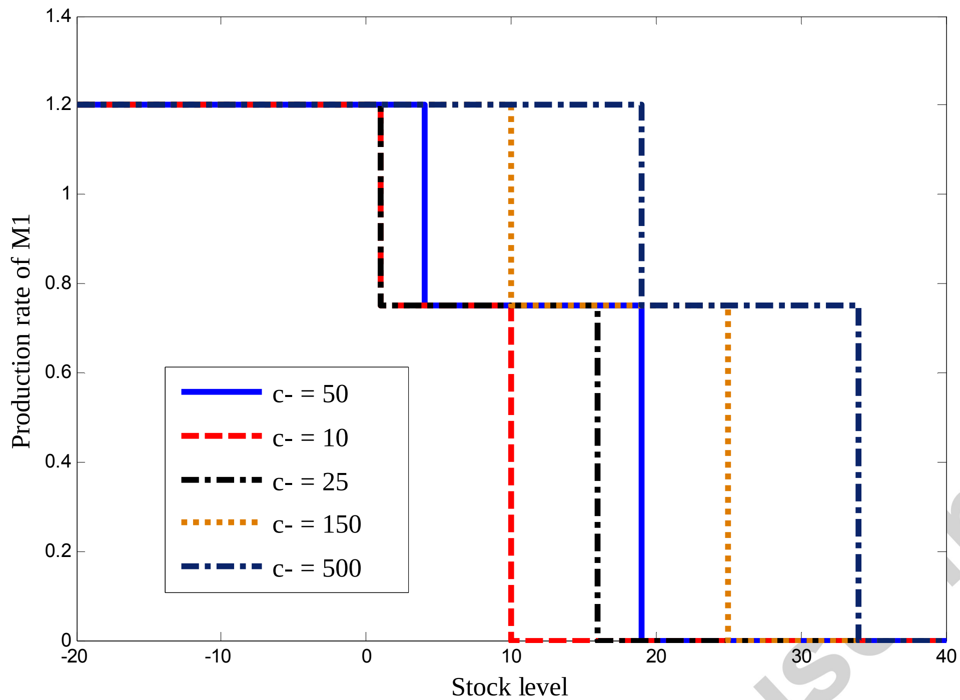
4.2 Sensitivity analyses

A set of numerical examples were considered to measure the sensitivity of the control policies obtained and to illustrate the contribution of this paper. The sensitivity of the control policies is analyzed according to the variation of the backlog costs and the machine parameters.

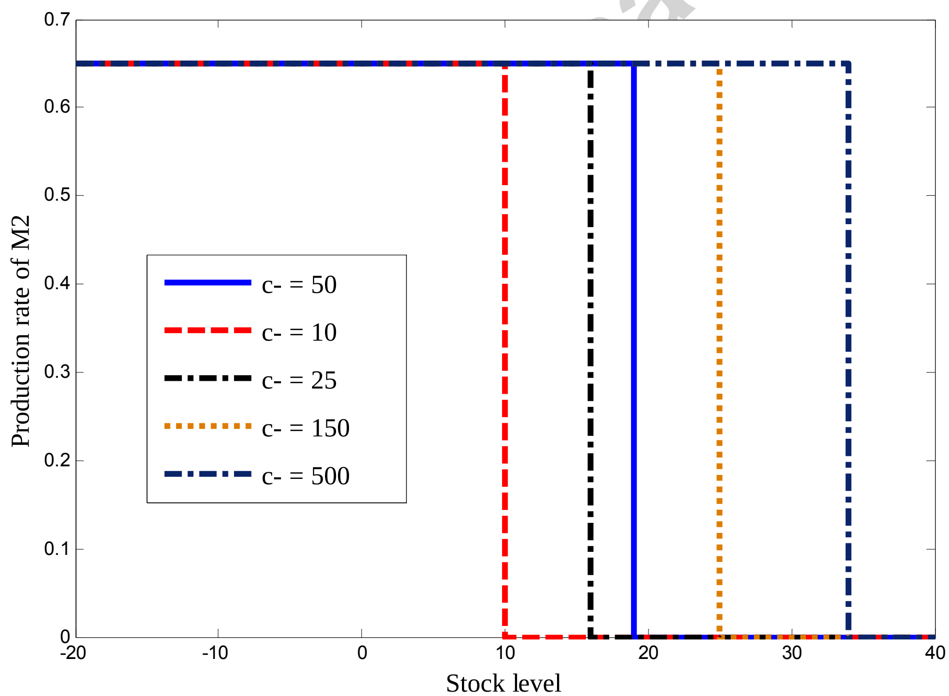
4.2.1 Sensitivity analysis with respect to backlog costs



a) Effect on the threshold values

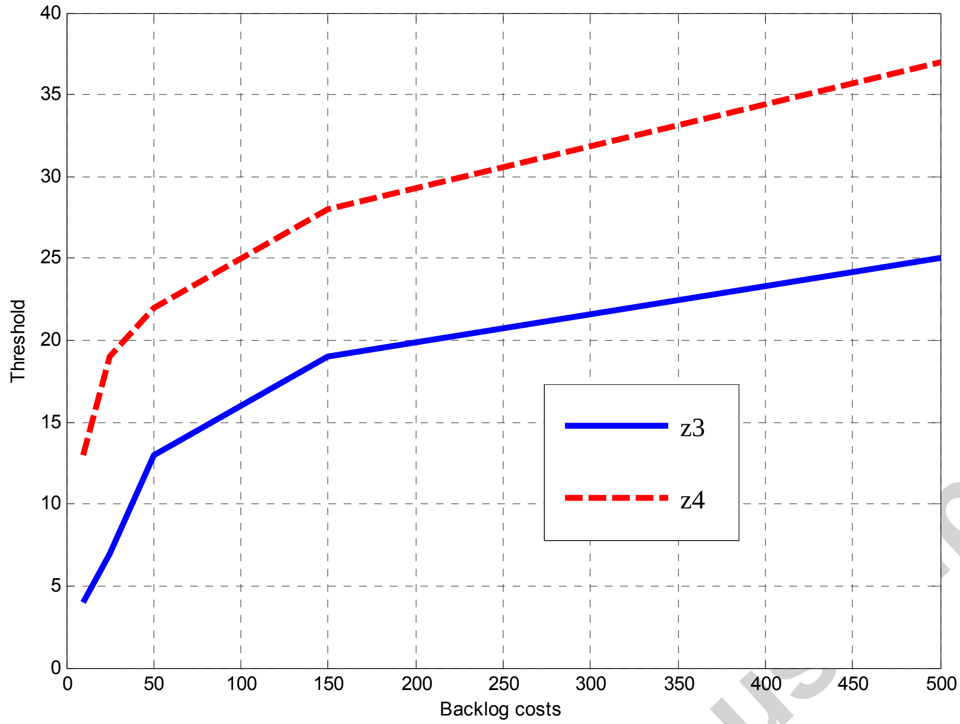


b) Effect on the production rate of M_1

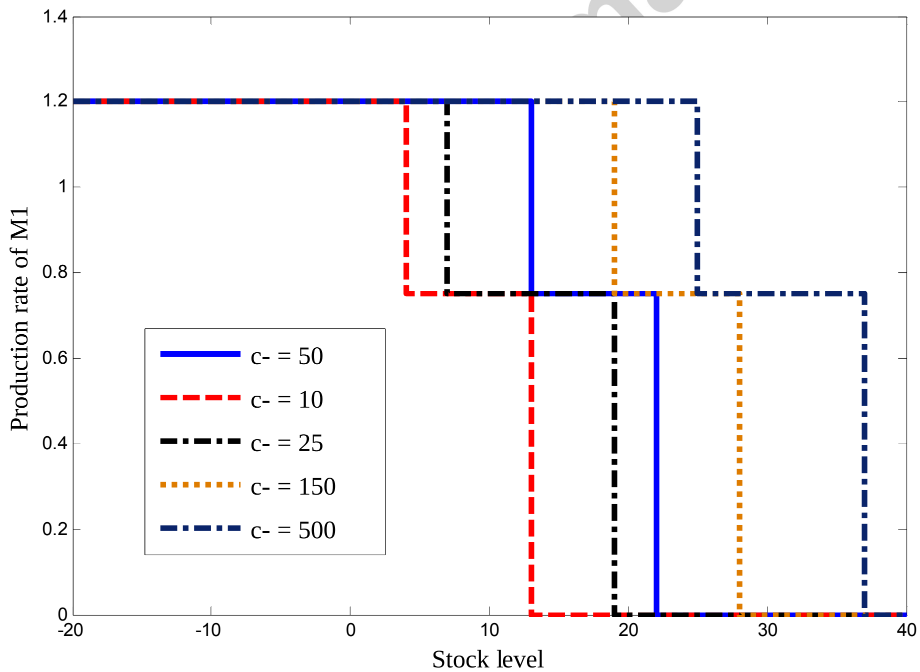


c) Effect on the production rate of M_2

Fig. 8: Sensitivity to the variation of backlog costs at mode 1



a) Effect on the threshold values



b) Effect on the production rate of M_1

Fig. 9: Sensitivity to the variation of backlog costs at mode 2

The results presented in Figs. 8 and 9 show the behavior of the productivities of machines according to variations of backlog costs. Based on these results, we can see that low backlog cost values

($0 \le c^{-} \le 25$) do not affect the threshold z_1. This is logical because at mode 1, when both machines are operational, the system does not use the first machine enough to its maximal productivity in order to take account of its reliability. The thresholds z_1, z_2, z_3 and z_4 increase as the backlog costs increase in order to avoid further backlog costs. Fig. 9 shows that the threshold values of M_1 at mode 2 (z_3 and z_4) are higher than the thresholds at mode 1 (z_1 and z_2) because the second machine is under repair. We therefore need a lot of parts in stock to avoid further backlog costs.

4.2.2 Sensitivity analysis with respect to machine parameters

This section analyzes the sensitivity of the threshold values with the respect to the parameters of the two machines, as shown in Figs. 10 to 17. The results show that the variation of the parameter q_{12}^{2} does not affect the thresholds z_1 and z_3 . This adequately reflects the phenomenon of degradation of our system. The productivity of M_1 should be reduced to its economical value when closing to a comfortable stock level in order to ensure its reliability. We recall that z_1 and z_3 are the first hedging point policies of M_1 at mode 1 and mode 2, respectively.

Let us now analyze the sensitivity of the thresholds according to each machine parameter.

a. Varying θ_1 (failure rate of M_1 for $u_1 \in (U, u_{1\max}]$)

When θ_1 increases, z_1 remains constant, z_3 decreases, and z_2 and z_4 increase. M_2 will necessarily tend to be more commonly used at mode 1 (both machines are producing) and M_1 will be at its economical productivity level ($u_1 = U$) at mode 2 (M_1 runs alone) in order to account for the reliability (the probability of failure at the maximum value is high). When θ_1 decreases, z_1 and z_3 increase because the probability of failure, for $u_1 \in (U, u_{\max}]$, is low. The other parameters of the control policy move as predicted, from a practical perspective (see Fig. 10).

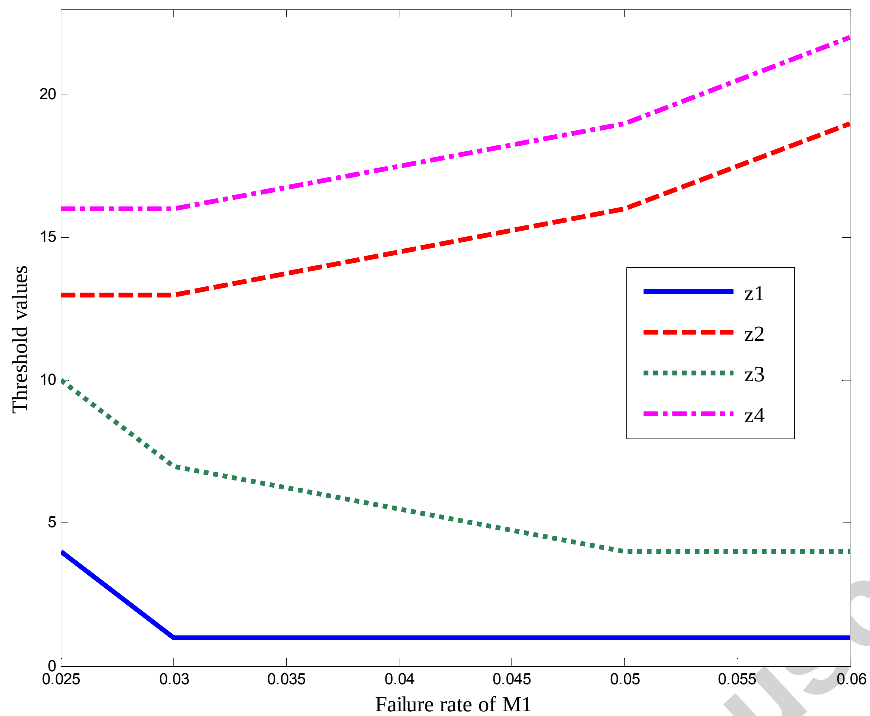


Fig. 10: Sensitivity analysis with respect to failure rate of M_1 for $u_1 \in (U, u_{1\max}]$

b. Varying θ_2 (failure rate of M_1 for $u_1 \in [0,U]$)

When θ_2 increases, the thresholds z_1 and z_3 increase, while z_2 and z_4 remain constant. This means that we must limit the use of M_1 at its economical productivity level because doing so increases the second failure rate; the threshold z_1 remains constant, as do the other parameters of the control policy, when θ_2 decreases (Fig. 11).

c. Varying q_{12}^2 (failure rate of M_2)

When q_{12}^2 decreases, the thresholds z_2 and z_4 decrease. This means that the system will stay at mode 1 for a long time before transitioning to mode 2 because the probability of failure of the second machine decreases. As for z_1 and z_3, their values remain constant. As a result, M_1 will tend to be used to its maximal productivity in order to avoid backlogs. The thresholds z_1, z_2, z_3 and z_4 remain constant when q_{12}^2 increases (see Fig. 12).

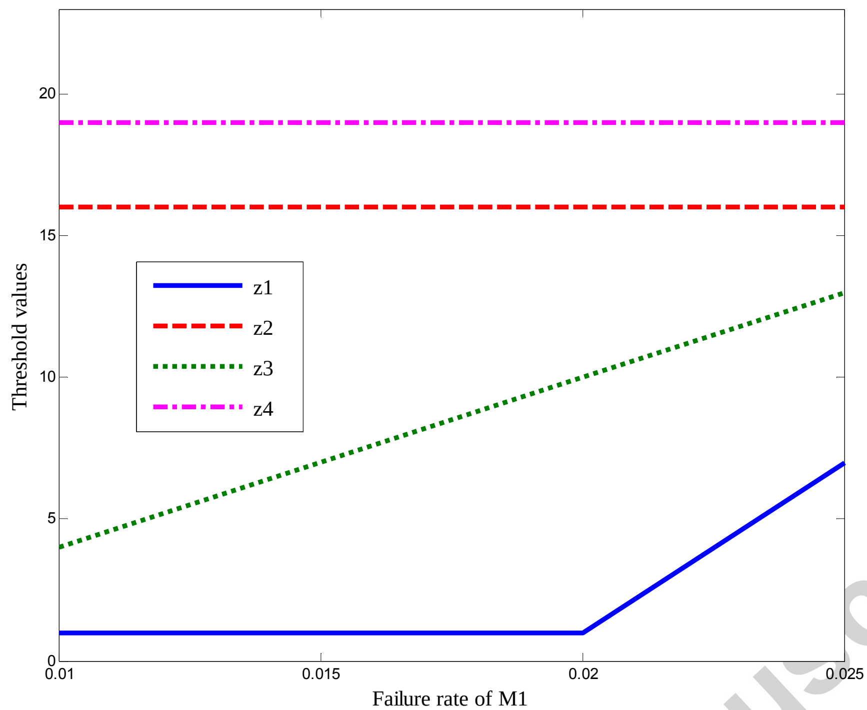


Fig. 11: Sensitivity analysis with respect to failure rate of M_1 for $u_1 \in [0, U]$

d. Varying q_{21}^1 (repair rate of M_1)

When q_{21}^1 increases, the thresholds z_1, z_2, z_3 and z_4 decrease in order to avoid over-stocking because the probability of repairing M_1 is high. There is a tendency to use M_1 and M_2 less when the repair rate of the main machine increases. If M_1 breaks down, it soon returns to the operational state. The parameters of the control policy move as predicted, from a practical perspective when q_{21}^1 decreases (Fig. 13).

e. Varying q_{21}^2 (repair rate of M_2)

The parameters z_1 and z_3 remain constant when q_{21}^2 increases; when q_{21}^2 decreases, z_1, z_2, z_3 and z_4 increase in order to avoid backlogs because the repair time of M_2 is long. If this machine fails, it will later return to its operational mode (Fig. 14).

f. Varying $u_{1\max}$ (maximal productivity of M_1)

The values of z_1 increases when $u_{1\max}$ increases. This inevitably increases the chances of M_1 being used to its maximal productivity at mode 1. At mode 2, where M_2 is under repair, z_3 decreases and z_4 remains constant. The productivity of M_1 must be reduced to its economical value to take account of its reliability. When $u_{1\max}$ decreases, the threshold z_1 decreases and the other parameters remain constant. This means M_1 must be used less to its maximum productivity (Fig. 15).

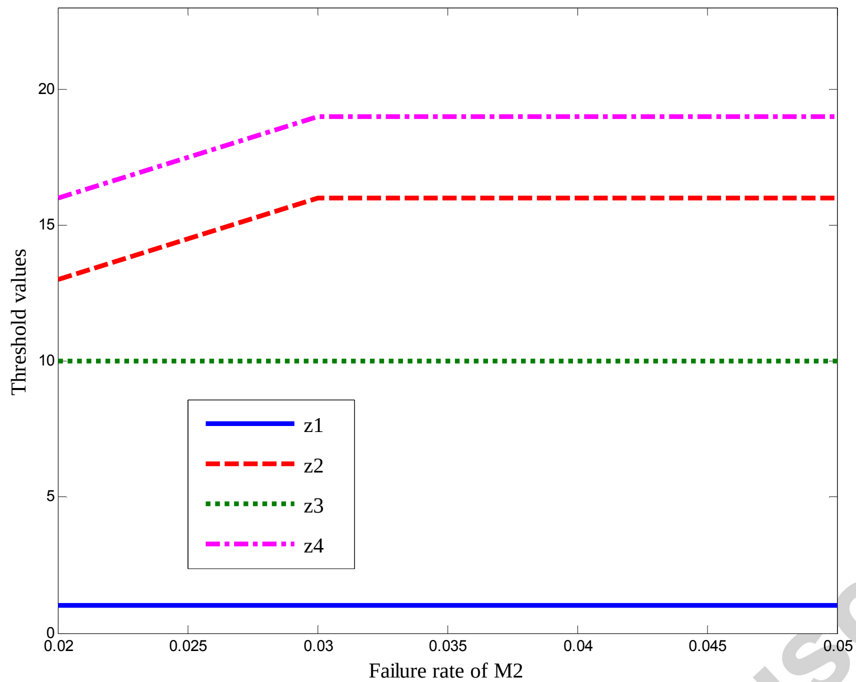


Fig.12: Sensitivity analysis with respect to failure rate of M_2

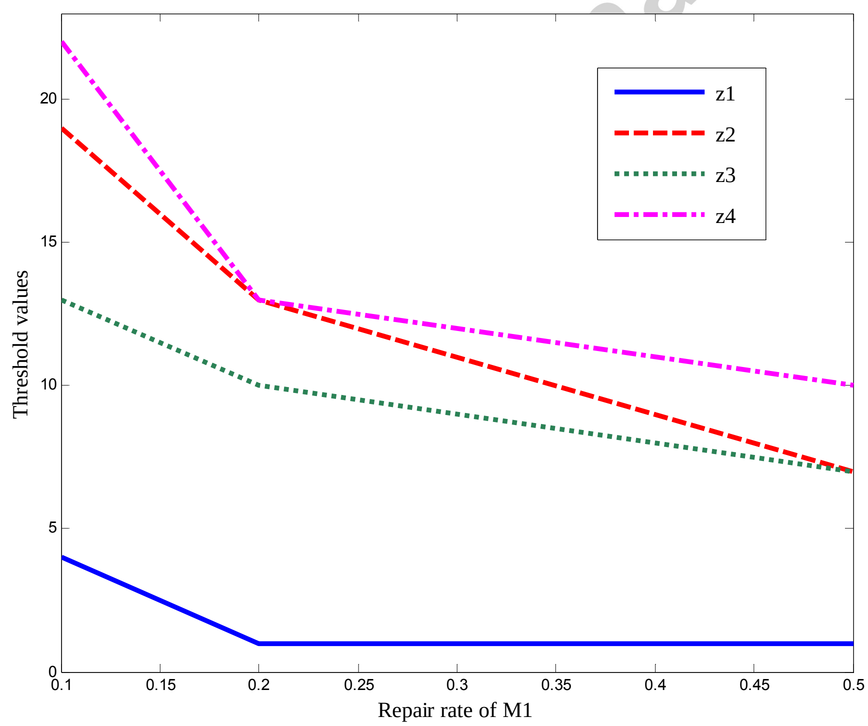


Fig. 13: Sensitivity analysis with respect to repair rate of M_1

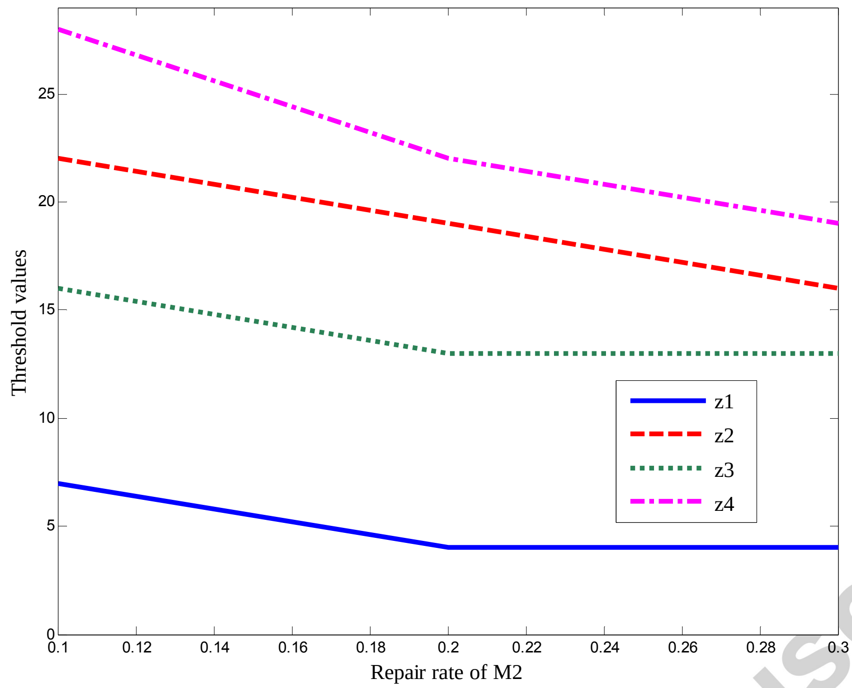


Fig. 14: Sensitivity analysis with respect to repair rate of M_2

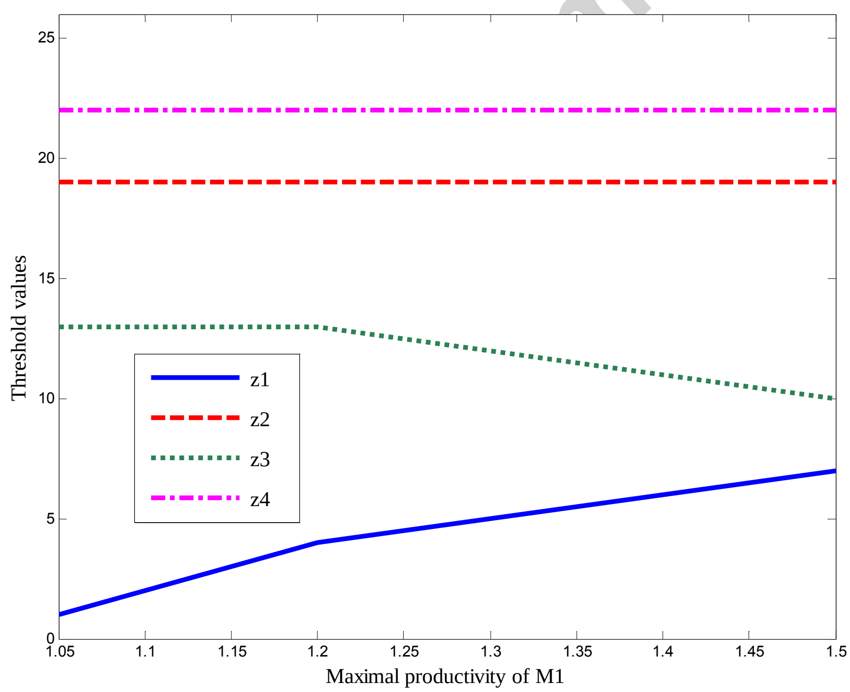


Fig. 15: Sensitivity analysis with respect to maximal productivity of M_1

g. Varying U (economical productivity of M_1)

The values of z_1 and z_3 increase and z_2 and z_4 remain constant when the economical productivity of M_1 decreases. This must increase the likelihood of M_1 being used to its maximum productivity at mode 1 and mode 2. The parameters of the control policy move as predicted, from a practical perspective when U decreases. See Fig. 16.

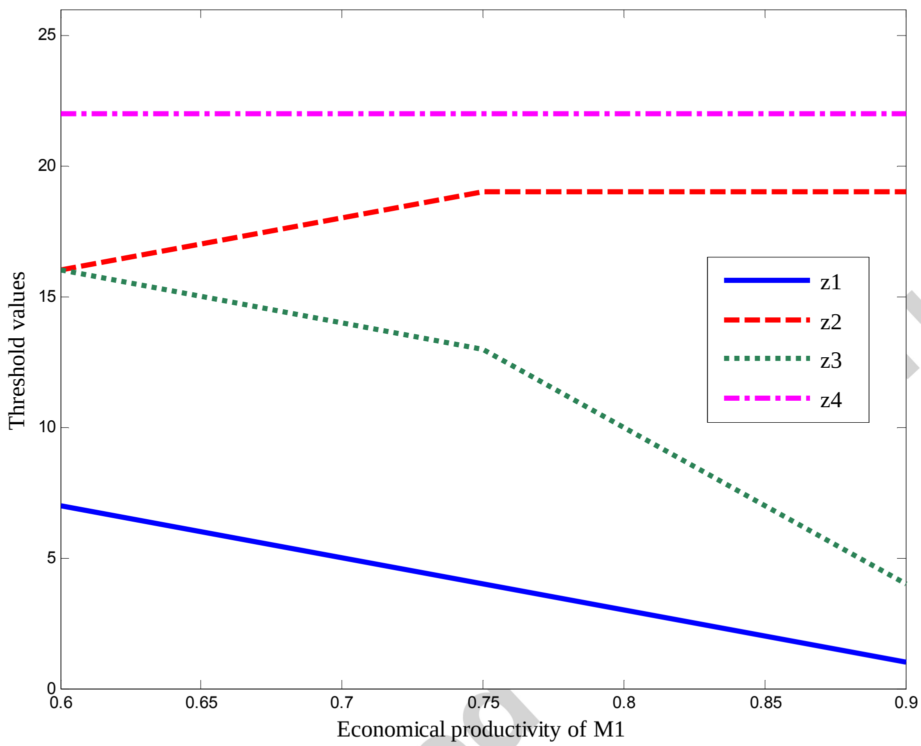


Fig. 16: Sensitivity analysis with respect to economical productivity of M_1

h. Varying $u_{2\max}$ (maximal productivity of M_2)

When $u_{2\max}$ increases, z_1 remains constant, and z_2, z_3 and z_4 decrease in order to avoid over-stocking. The parameters of the control policy move as predicted, from a practical perspective when $u_{2\max}$ decreases, in order to avoid over-shortages (Fig. 17).

Through the observations drawn made from the sensitivity analysis, it clearly appears that the results obtained make sense, and confirm and validate the proposed approach. They show the usefulness of the proposed model, given that the parameters of the control policies move as predicted, from a practical perspective. The next section studies the case of production rate-dependent multiple failure rates.

5. Extensions to the case of multiple failure rates

5.1 Numerical example

Section 4 showed that the hedging point policies are optimal for two failure rates of the main machine. In this section, we study the case of manufacturing systems consisting of two machines with five failure rates depending on the productivity of the main machine. These failure rates are given as follows:

$$q_{12}^{1}(u_1)=\begin{cases}\theta_1 & \text{if } u_1 \le U_1\\ \theta_2 & \text{if } u_1 \in (U_1, U_2]\\ \dots & \\ \theta_5 & \text{if } u_1 \in (U_4, u_{1\max}]\end{cases}$$

(15)

where $0<\theta_1<\theta_2<\ldots<\theta_5$ and $0<U_1<U_2<U_3<U_4<U_5=u_{1\max}$.

The failure rate in Eq. (15) has the general form considered in Liberopoulos and Caramanis (1994),

$$q_{12}^{1}(u_1)=a\left(\frac{u_1}{u_{1\max}}\right)^b$$

(16)

where a and b are non-negative constants.

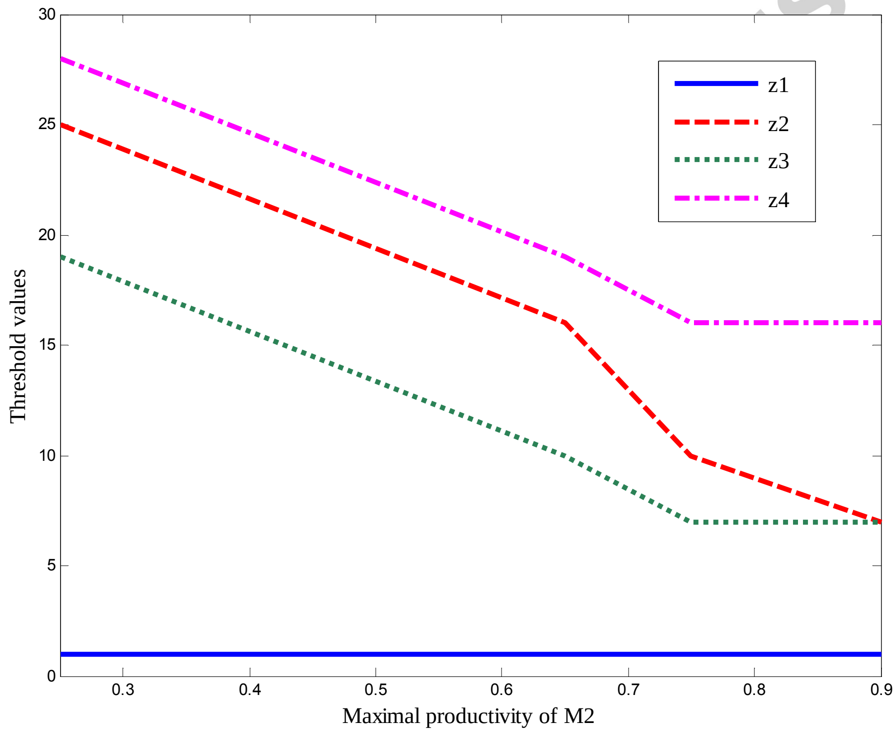


Fig. 17: Sensitivity analysis with respect to maximal productivity of M_2

The results for $a=0.02$ and different values of b are plotted in Fig. 18.

The curves plotted in Fig. 18 illustrate the impact of the machine's productivity on its dynamics. The solid sections represent the feasible productivity values (values for which the condition to meet customer demand is satisfied) when both machines are operational. The dashed sections represent the unfeasible values. The concave curve is represented by $b<1$ and the convex curve by $b>1$.

For a single machine, single product manufacturing system, where the failure rate depends on the production rate, it was concluded that an optimal feedback policy control does not exist if $U_n < d$, $n = 1,2,3,4,5$ (see Liberopoulos and Caramanis (1994); Martinelli (2010)). For a manufacturing system consisting of two machines, with a single product, such as the one studied in this paper, we examine the case of the main machine's productivity lower than the demand rate ($\frac{u_1}{u_{1\max}} < 0.9$). Typical results for productivity with values of $b = 0.4$ and $b = 3$ are shown in Figs. 19 and 20, respectively. The values used for $\frac{u_1}{u_{1\max}}, U_n$ and θ_n are presented in Table 3.

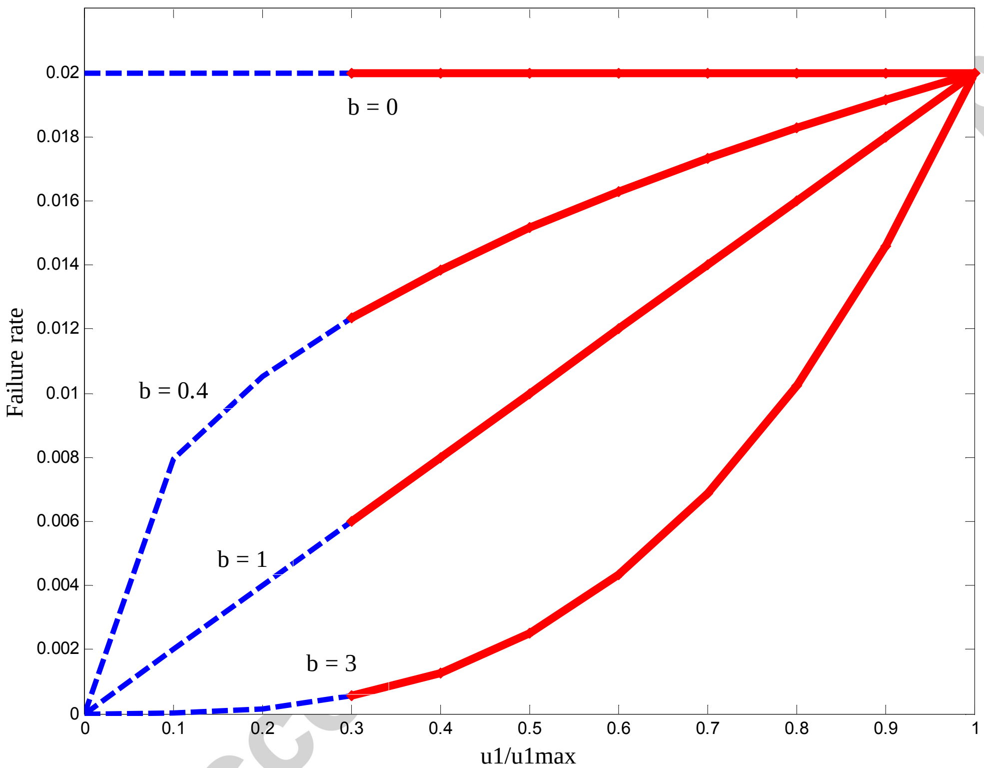


Fig. 18: Failure rate of the main machine

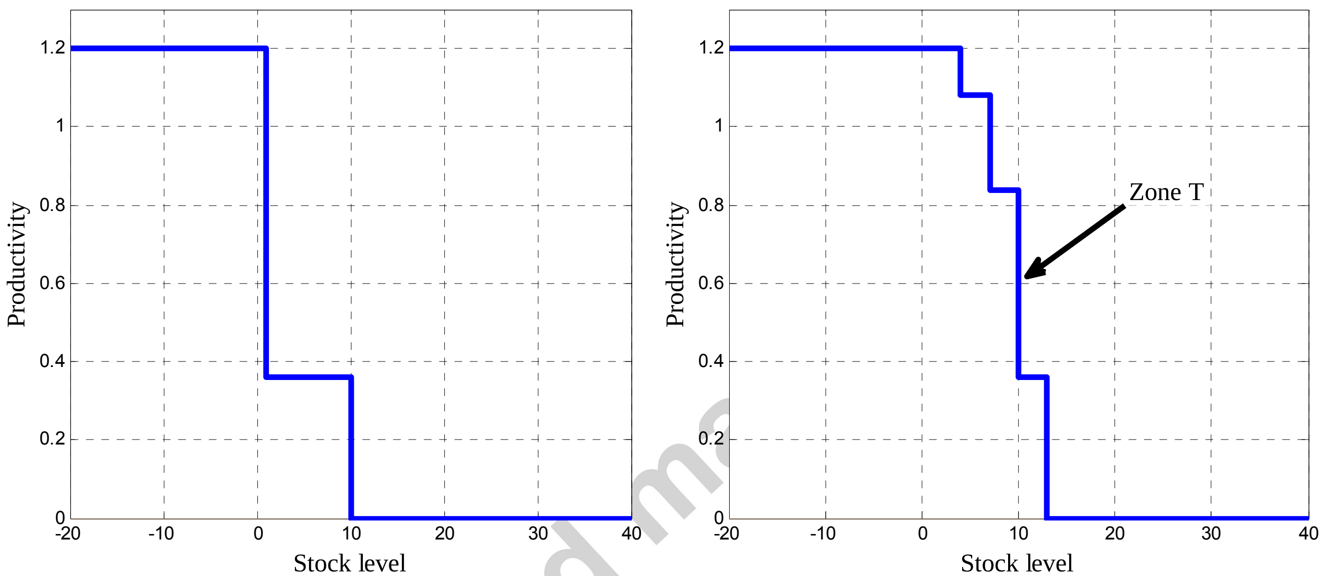
Table 3. Parameter values

$\frac{u_1}{u_{1\max}}$	0.3	0.5	0.7	0.9	1
U_n	0.36	0.60	0.84	1.08	1.2
$\theta_n(b=0.4)$	0.01236	0.01516	0.01734	0.01917	0.0200
$\theta_n(b=3)$	0.00054	0.0025	0.00686	0.01458	0.0200

Based on the results presented in Figs. 19 and 20, the productivity policy of M_1 defines three control rules at mode 1 (Figs. 19a and 20a) and four (Fig. 19b) or five (Fig. 20b) control rules at mode 2. More specifically, these rules state that:

i) When the stock level is higher than the optimal threshold point, M_1 does not produce.

ii) If the stock level is lower than the first threshold point ($y_{11}, y_{12}, z_{11}, z_{12}$), M_1 should be set to its maximal productivity. Note that the requirement $d < u_1$ is not imposed for the machine M_1 because the system has a supporting machine M_2 (both machines can satisfy the customer demand together).



a) Mode 1 b) Mode 2

Fig. 19: Productivity of M_1 at mode 1 and mode 2, $b = 0.4$

According to Fig. 19, the corresponding multiple threshold point policy has the structure of Eqs (17) and (18) for mode 1 and mode 2, respectively.

$$u_1(x,1) = \begin{cases} u_{1\max} & \text{if } x < y_{11} \\ U_1 & \text{if } y_{11} \le x < y_{21} \\ 0 & \text{if } x > y_{21} \end{cases}$$

(17)

where y_{11} and y_{21} are the first and second threshold values of M_1 at mode 1, respectively.

$$u_1(x,2) = \begin{cases} u_{1\max} & \text{if } x < y_{12} \\ U_4 & \text{if } y_{12} \le x < y_{22} \\ U_3 & \text{if } y_{22} \le x < y_{32} \\ U_1 & \text{if } y_{32} \le x < y_{42} \\ 0 & \text{if } x > y_{42} \end{cases}$$

(18)

where $y_{i2}, i = 1,2,3,4$ is the ith threshold value of M_1 at mode 2.

The optimal policy of Fig. 20 is defined by Eqs. (19) and (20) for mode 1 and mode 2, respectively.

$$u_1(x,1)=\begin{cases} u_{1\max} & \text{if } x<z_{11} \\ U_1 & \text{if } z_{11}\le x<z_{21} \\ 0 & \text{if } x>z_{21} \end{cases}$$

(19)

where z_{11} and z_{21} are the first and second threshold values of M_1 at mode 1, respectively.

$$u_1(x,2)=\begin{cases} u_{1\max} & \text{if } x<z_{12} \\ U_4 & \text{if } z_{12}\le x<z_{22} \\ U_3 & \text{if } z_{22}\le x<z_{32} \\ U_2 & \text{if } z_{32}\le x<z_{42} \\ U_1 & \text{if } z_{42}\le x<z_{52} \\ 0 & \text{if } x>z_{52} \end{cases}$$

(20)

where $z_{i2}, i=1,2,3,4,5$ is the ith threshold value of M_1 at mode 2.

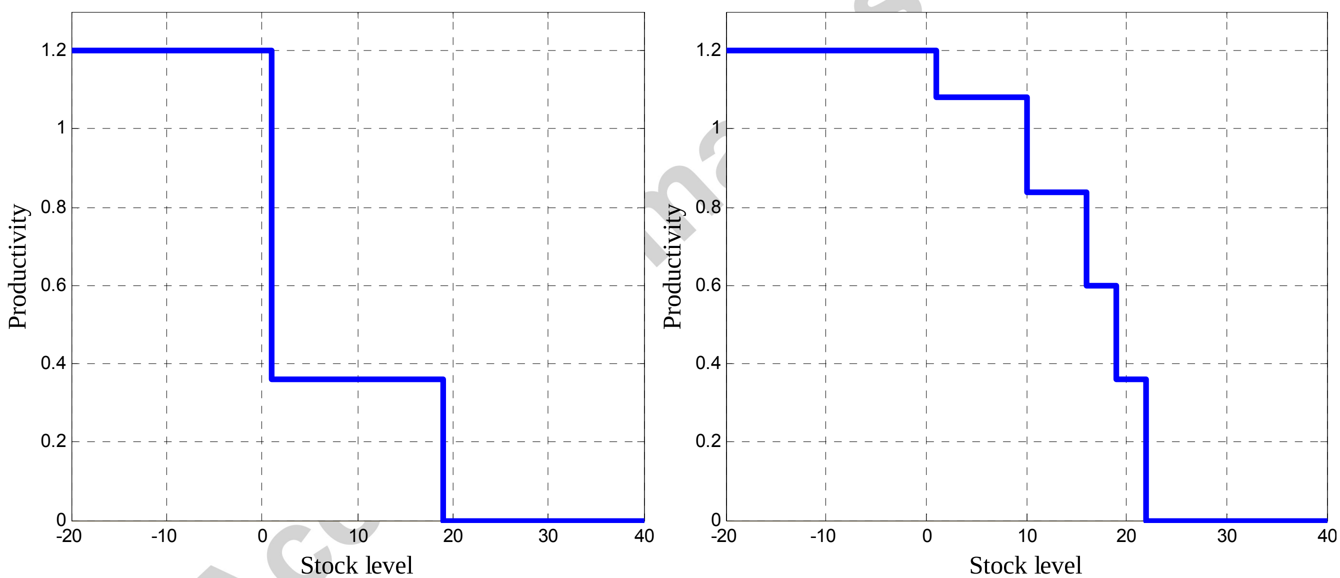


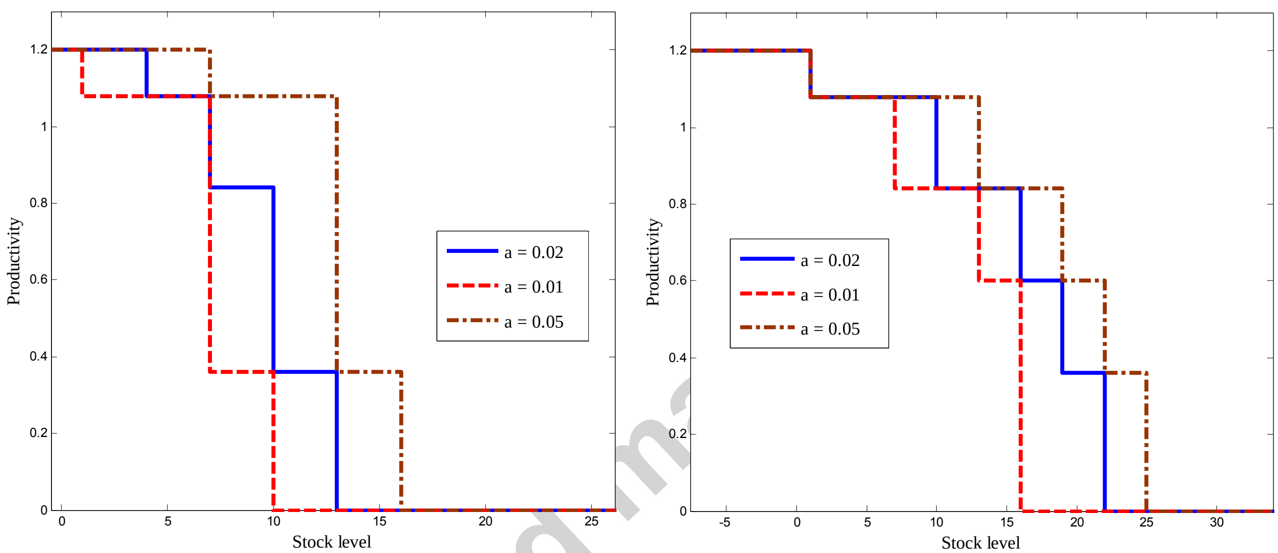
Fig. 20: Productivity of M_1 at mode 1 and mode 2, $b=3$

It clearly appears that at mode 2 (M_1 produces alone), when $u_1=U_3=0.84$ (meaning that the machine begins to produce at a rate lower than the demand rate), the system switches directly from $U_3=0.84$ to its minimal value $U_1=0.36$ (see Zone T in Fig. 19b). This is logical because the system has to avoid shortages and ensure its reliability at the same time.

5.2 Sensitivity analyses

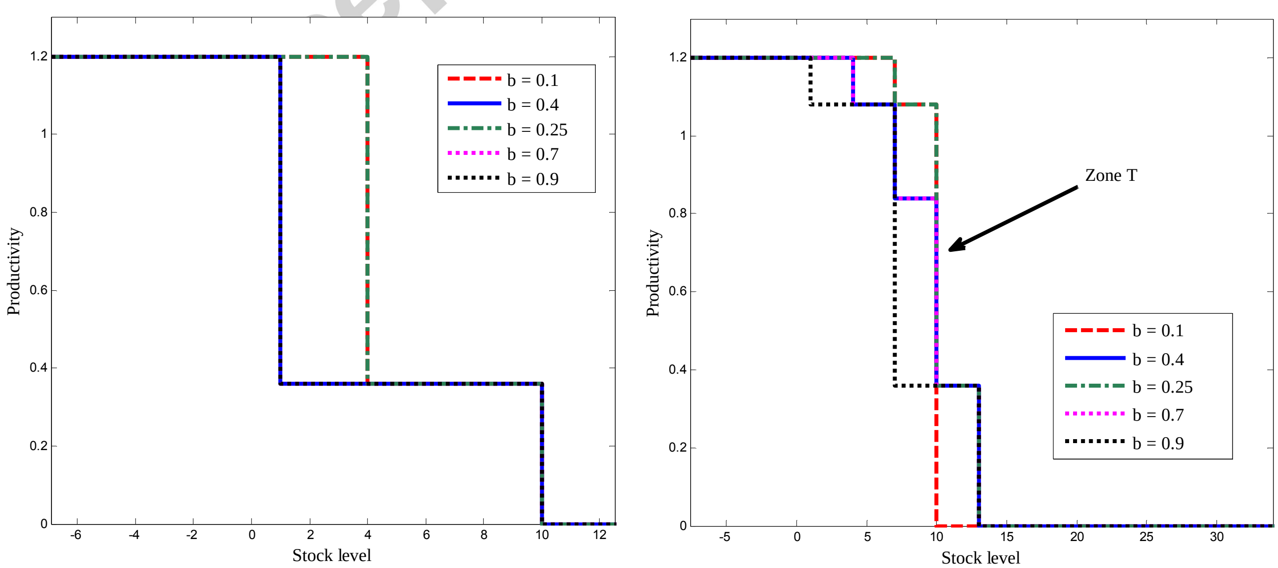
This section explains the usefulness of the obtained control policy. We perform a sensitivity analysis according to the variation of the parameters "a", "b" and "n" to illustrate the contribution of the proposed approach, and also to confirm the structure of the control policy. The productivity is presented in Figs. 21 to 24.

The effect of the variation of the parameter a on the productivity policy is illustrated in Fig. 21. The parameter takes three values: $a = 0.01$, 0.02 and 0.05. When the parameter is low, $a = 0.01$, it means the system experiences fewer failures. Thus, the threshold value is low. If the parameter is set to $a = 0.05$, the system needs more protection against failures, leading to an even greater increase in the threshold value. It is worth mentioning that when a increases, the probability of failure of the system increases and the reliability of the machine is reduced. This necessarily leads to a high likelihood of increasing the threshold level in order to avoid shortages.



a) Concave case "$b = 0.4$" b) Convex case "$b = 2$"

Fig. 21: Sensitivity to the variation of "a"

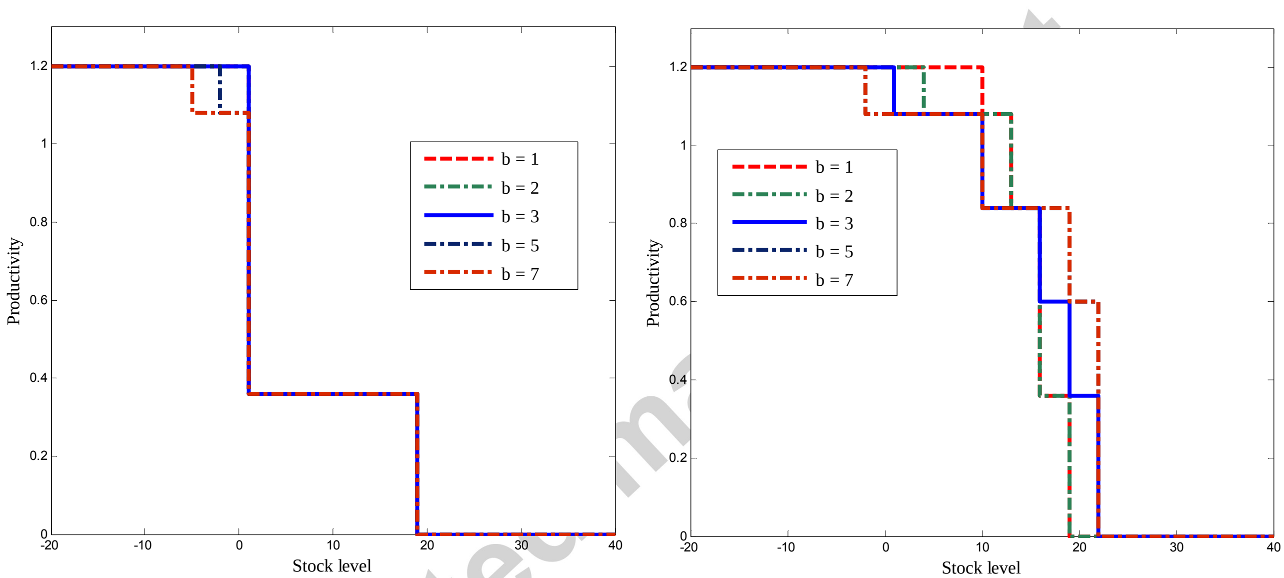


a) Mode 1 b) Mode 2

Fig. 22: Sensitivity to the variation of "b"; Concave case

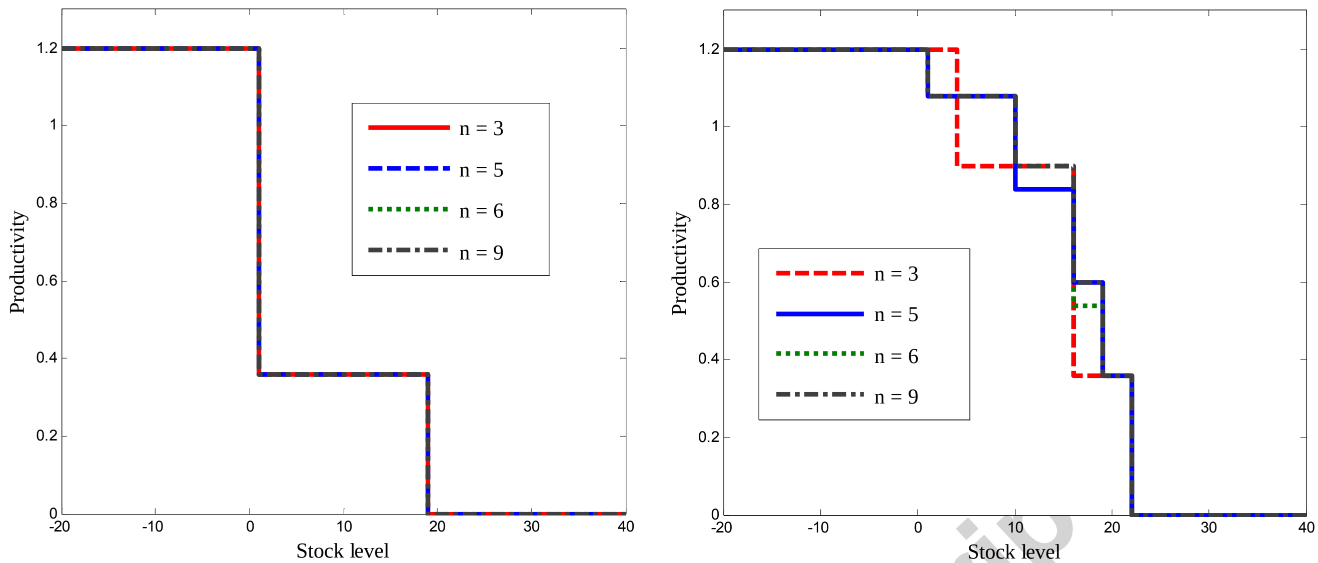
From the results obtained in Figs. 22 and 23, we notice that when parameter b increases, the threshold values increase. If parameter b increases, for the same productivity value, the failure rate decreases. This means that the system produces for a long time before failure. The variation of parameter b does not affect the second threshold value at mode 1.

At mode 2, when b decreases ($b = 0.1$ and $b = 0.25$), zone T increases and is the same (see Fig. 22b). Decreasing b in the concave case means that the failure rate increases. The system must store to avoid shortages.



a) Mode 1 b) Mode 2

Fig. 23: Sensitivity to the variation of "b"; Convex case



a) Mode 1 b) Mode 2

Fig. 24: Sensitivity to the variation of "n"

The results of Fig. 23 show that in the convex case, decreasing parameter b does not affect the productivity trend at mode 1. However, when b increases ($b = 5$ and b = 7), Fig. 23a shows three stages instead of two stages, as in the basic case ($b = 3$). It is clear that higher values of b reduce the deterioration of the system. The system can produce to its intermediate speeds before reaching the minimal productivity.

The results of Fig. 24a show that the variation of parameter n does not affect the threshold values at mode 1. At mode 2 (Fig. 24b), the number of stages does not change when n increases. For example, when $n > 5$, we have five threshold parameters, such as in the case of five failure rates. However, when the number of failures decreases, the number of stages decreases. The next section presents the discussions and how to implement the obtained control policies.

6. Discussions and policies implementation

The results of Figs. 4 (M_1 at mode 1) and 6 (M_1 at mode 2) confirm the possible practical suggestion based on the analysis of Fig. 1. The results suggest that to obtain gains in availability of the main machine and to reduce the total machining cost incurred, it may be beneficial to decrease the productivity speed from the maximal value to the economical value when the inventory level approaches the maximal threshold values.

In the sensitivity analysis, we observe that the threshold values increase as the backlog costs increase (Figs. 8 and 9). This seems natural in order to avoid further backlog costs. Figs. 10 to 17 show that the parameters of the control policy move as predicted, from a practical perspective, when the machine parameters change.

For the manufacturing system considered, the optimal control policies are characterized by four different threshold parameters (z_1, z_2, z_3 and z_4) for two failure rates of M_1, which constitute a main finding of this paper. For n different failure rates of the main machine, the control policies will depend on more than n different threshold parameters. In Section 5, the case of five failure rates was studied. The most important observation from the results is that the optimal policy exists and is still equivalent to the multiple threshold policy. In the concave case, the structure of the productivity policy is the multiple hedging point policy because the system consists of two machines. The effect of the second machine reduces the concavity of the curve. Therefore, the concave curve is close to linear ($b = 1$), or is nearly convex. At mode 2, where M_2 is under repair, we have multiple thresholds in both cases. With two machines, even if one machine fails, the system knows that it exists and will return to the operational state in a relatively short time.

At mode 1, where both machines produce, Figs. 19a and 20a show that the optimal control policy of the main machine is characterized by two threshold parameters, such as in the system described in Section 4 with two failure rates. In the case of five failure rates, the system prefers to skip the intermediate productivities of the main machine and to produce directly to the minimal value ($U_1 = 0.36$) over a long period. It can then use the supporting machine to fill the customer demand. This is logical because the probability of failure of the main machine increases with high productivity, while the failure rate of the supporting machine is constant. However, at mode 2, the optimal policy is characterized by four (concave case) or five (convex case) threshold parameters (Figs. 19b and 20b). The number of stages in the concave case is less than the number of stages in the convex case because from $U_3 = 0.84$ to $U_2 = 0.60$, Table 3 shows that the differences between the failure rates are low. In contrast, this difference is higher in the convex case. The system must rapidly reduce the productivity of the machine to account for its reliability. Hence, the system passes through $U_2 = 0.60$ before reaching $U_1 = 0.36$ (see Fig. 20b). Another remark regarding Fig. 20b is that $z_{52} = 22 = z_4$. The threshold parameter is the same with five failures. This means that to achieve gains in availability of the main machine and to reduce the total machining costs incurred, the system does several speed jumps before reaching the optimal stock level.

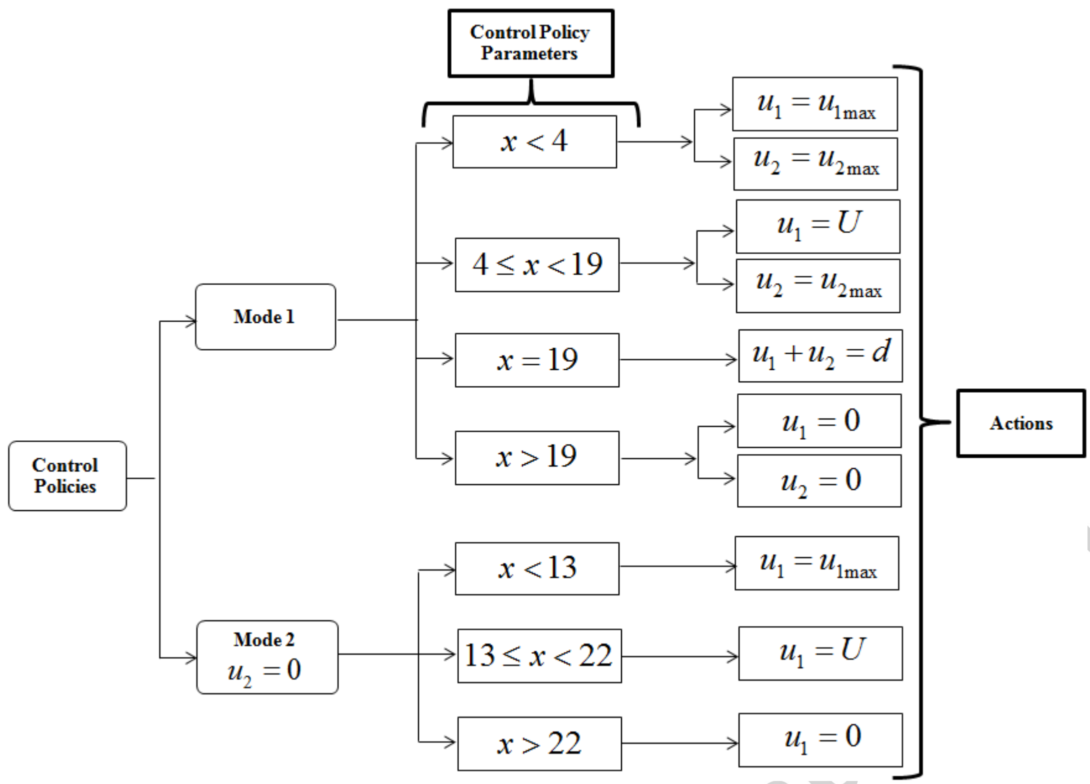


Fig. 25: Illustration of the production policy

Through the sensitivity analysis conducted, Fig. 21 shows that the threshold values increase when the parameter a increases. This necessarily leads to a sustained increase in the threshold level in order to avoid shortages. According to Figs. 22 and 23, the results show that the parameters of the control policy move as predicted, from a practical perspective, when parameter b changes. Unlike in the concave case, where Fig. 22a shows two stages, as in the basic case, Fig. 23a shows three stages when b increases. This is due to the fact that the failure rate decreases faster in the convex case than it does in the concave case. We should recall that the failure rate decreases when b increases. The system can use its intermediate speeds before achieving minimal productivity. The results of Fig. 24a show that at mode 1 where both machines are operational, for gaining in availability of the main machine, the system maintains the same threshold values and uses the supporting machine to fill the customer demand. Increasing of parameter n ($n > 5$) does not change the number of stages at mode 2 (see Fig. 24b) because for several values of productivity comprised between $U_1 = 0.36$ and $u_{1\max} = 1.2$, the differences between values are too low. In this way, the system skips some intermediate values. If the value of $u_{1\max}$ increases, the number of threshold parameters will be increased as well. However, when $n = 3$, there is three threshold parameters. The relevance of the sensitivity analysis is apparent, since it seems that our results are logical and consistent, and this enables us to confirm the structure of the control policies obtained.

Fig. 25 illustrates the implementation of the control policy when the number of failure rates is $n = 2$. This illustration shows the actions that should be taken by the manager when both machines are producing (mode 1), and when the supporting is under repair (mode 2). Based on the diagram of Fig. 25, we can see how the production speed of the main machine is set to different values depending on the both machine modes (functioning or failure) and the stock level. Thus, the obtained policies have a direct managerial implication, namely the manager can use obtained results to define the parameters of the manufacturing system in order to optimize the production process.

7. Conclusion

The number of scientific publications in the field of deteriorated systems is growing steadily, reflecting the increasing importance of this subject. However, the reported works are mostly based on systems which deteriorate with the age and the number of failures. This paper investigates the problem of minimizing a cost function which penalizes both the presence of waiting customers and the inventory surplus. The manufacturing system studied comprises parallel machines subject to a non-homogeneous Markov process, with the failure rate depending on the productivity. The machines produce a single part type. We developed the stochastic optimization model of the considered problem with two decision variables (productivities of the main and the supporting machines) and one state variable (stock level of final products). From the numerical study, it has been found that for two parallel machines systems, when the failure rate of the main machine depends on its productivity, the hedging point policies are optimal within a four-threshold feedback policy, and the reliability of the machines is enhanced. The results also show that to reduce the total machining cost, it may be beneficial to decrease the productivity of the main machine from its maximal value to its economical value when the inventory level approaches the threshold value. We illustrated and validated the proposed approach using a numerical example and a sensitivity analysis. We have studied the case of manufacturing systems involving multiple failure rates, and the results obtained are very satisfactory and may be productive for future research to address the issue of multiple-part-type, random demand rates and multiple-machine (more than two machines) systems.

Appendix. Numerical approach

To solve the HJB equations, we used a numerical method based on the Kushner (1992) approach, such as in Gharbi et al. (2011). By approximating $v(\alpha, x)$ by a function $v^h(\alpha, x)$ and the first-order partial derivative of the value function $\frac{\partial v(\alpha, x)}{\partial x}$ by:

$$\frac{\partial v(x,\alpha)}{\partial x} = \begin{cases} \frac{1}{h}\left(v^h(\alpha, x+h) - v^h(\alpha, x)\right) & \text{if } (u_1 + u_2 - d) > 0 \\ \frac{1}{h}\left(v^h(\alpha, x) - v^h(\alpha, x-h)\right) & \text{otherwise} \end{cases}$$

The HJB equation becomes:

$$v^h(x,\alpha) = \min_{(u_1,u_2)\in \mathrm{A}(\alpha)} \left[\frac{ g(x,\alpha) + \sum_{\beta\neq\alpha} \lambda_{\alpha\beta} v^h(x,\beta) + \frac{(u_1+u_2-d)}{h} \left[\begin{array}{l} v^h(x+h,\alpha) Ind\{u_1+u_2-d \geq 0\} \\ + v^h(x-h,\alpha) Ind\{u_1+u_2-d < 0\} \end{array} \right] }{ \left(\rho + \frac{|u_1+u_2-d|}{h} + |\lambda_{\alpha\alpha}| \right) } \right]$$

(A.1)

with $\lambda_{\alpha\alpha} = -\sum_{\beta\neq\alpha} \lambda_{\alpha\beta}$, $\mathrm{A}^h(\alpha)$ is the numerical control grid and $Ind\{\Phi\} = \begin{cases} 1 & if \quad \Phi \text{ is true} \\ 0 & otherwise \end{cases}$

The system of Eq. (A.1) can be interpreted as the infinite horizon dynamic programming equation of a discrete-time, discrete-state decision process, as in Boukas and Haurie (1990). In this paper, we use the value iteration procedure to approximate the value function given by Eq. (A.1). Dehayem et al. (2011) and references therein provide details on such methods.

The discrete dynamic programming Eq. (A.1) gives the following six equations:

- **mode 1**

$$v^h(x,1) = \begin{cases} V_1^h(x,1) & \text{if } u_1 \in (U, u_{1\max}] \\ V_2^h(x,1) & \text{if } u_1 \in [0, U] \end{cases} \qquad \text{with}$$

(A.2)

$$V_1^h(x,1) = \min_{\substack{u_1 \in (U, u_{1\max}] \\ u_2 \in [0, u_{2\max}]}} \left[\frac{g(x,\alpha) + \frac{(u_1+u_2-d)}{h} \begin{bmatrix} v^h(x+h,1) Ind\{u_1+u_2-d \geq 0\} + \\ v^h(x-h,1) Ind\{u_1+u_2-d < 0\} \end{bmatrix} + q_{12}^2 v^h(x,2) + \theta_1 v^h(x,3)}{\left(\rho + \frac{|u_1+u_2-d|}{h} + q_{12}^2 + \theta_1\right)} \right]$$

$$V_2^h(x,1) = \min_{\substack{u_1 \in [0, U] \\ u_2 \in [0, u_{2\max}]}} \left[\frac{g(x,\alpha) + \frac{(u_1+u_2-d)}{h} \begin{bmatrix} v^h(x+h,1) Ind\{u_1+u_2-d \geq 0\} + \\ v^h(x-h,1) Ind\{u_1+u_2-d < 0\} \end{bmatrix} + q_{12}^2 v^h(x,2) + \theta_2 v^h(x,3)}{\left(\rho + \frac{|u_1+u_2-d|}{h} + q_{12}^2 + \theta_2\right)} \right]$$

- **mode 2**

$$v^h(x,2) = \begin{cases} V_1^h(x,2) & \text{if } u_1 \in (U, u_{1\max}] \\ V_2^h(x,2) & \text{if } u_1 \in [0, U] \end{cases} \qquad \text{with}$$

(A.3)

$$V_1^h(x,2) = \min_{u_1 \in (U, u_{1\max}]} \left[\frac{g(x,\alpha) + \frac{(u_1-d)}{h} \left[v^h(x+h,2) Ind\{u_1-d \geq 0\} + v^h(x-h,2) Ind\{u_1-d < 0\} \right] + q_{21}^2 v^h(x,1) + \theta_1 v^h(x,4)}{\left(\rho + \frac{|u_1-d|}{h} + q_{21}^2 + \theta_1\right)} \right]$$

$$V_2^h(x,2) = \min_{u_1 \in [0, U]} \left[\frac{g(x,\alpha) + \frac{(u_1-d)}{h} \left[v^h(x+h,2) Ind\{u_1-d \geq 0\} + v^h(x-h,2) Ind\{u_1-d < 0\} \right] + q_{21}^2 v^h(x,1) + \theta_2 v^h(x,4)}{\left(\rho + \frac{|u_1-d|}{h} + q_{21}^2 + \theta_2\right)} \right]$$

- **mode 3**

$$v^h(x,3) = \min_{u_2 \in [0, u_{2\max}]} \left[\frac{ \begin{array}{l} g(x,\alpha) + \dfrac{(u_2 - d)}{h}\left[v^h(x+h,3) Ind\{u_2 - d \geq 0\} + v^h(x-h,3) Ind\{u_2 - d < 0\}\right] + \\ q_{21}^1 v^h(x,1) + q_{12}^2 v^h(x,4) \end{array} }{ \left(\rho + \dfrac{|u_2 - d|}{h} + q_{21}^1 + q_{12}^2 \right) } \right]$$

(A.4)

- mode 4

$$v^h(x,4) = \min \left[\frac{ g(x,\alpha) + q_{21}^1 v^h(x,2) + q_{21}^2 v^h(x,3) + \dfrac{d}{h} v^h(x-h,4) }{ \left(\rho + \dfrac{d}{h} + q_{21}^1 + q_{21}^2 \right) } \right]$$

(A.5)

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