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Hierarchical control of production with stochastic demand in manufacturing systems

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Abstract

This paper addresses a stochastic optimal control problem for a reliable single-product manufacturing system with a finite capacity. The demand by customers is stochastic in a finite planning horizon, and is described by a known continuous function. A two-level hierarchical control model is developed. In the first level, a stochastic linear-quadratic optimal control problem is formulated to determine the target values of the state and control variables, with the variables being the desired inventory level and desired production rate, respectively. The goal of the second level is to maintain the inventory as close as possible to its desired level as well as to maintain the production rate as close as possible to its desired rate with fluctuating demand. This problem is represented as a recursive optimization control problem (predictive control) with one state variable (the inventory level) and one control variable (the production rate). The purpose of this study is to establish the optimal production control strategies satisfying a stochastic demand that minimizes the total average quadratic cost and the sum of the mean square deviations of inventory and production for the first and second levels, respectively. The solution is obtained by applying the stochastic optimal control principle based on Pontryagin's maximum principle using the Euler-Maruyama scheme. A numerical example and sensitivity analyses are presented to illustrate the usefulness of the proposed approach.

Keywords: Manufacturing systems; Stochastic optimal control; Stochastic demand; Pontryagin's maximum principle; Euler-Maruyama scheme; Shooting methods.

1. Introduction

Generally, a manufacturing system inventory is very important as it allows the company to respond more quickly to customer demand, given that the demand rate of manufacturing produced parts is usually stochastic. In this study, the system considered is composed of one machine producing a single type of product. A stochastic production planning problem can be formulated as a stochastic optimal control problem, which involves finding the optimal production strategy to be used in managing the inventory of the system and satisfy the customer's stochastic demand over the planning horizon.

The last two decades have seen a jump in growing interest among researchers in the application of the stochastic optimal control theory in different fields. Initially used in engineering and other fields of applied mathematics, the stochastic optimal control theory is also generally used in finance, economics, medicine, maintenance and production planning to solve a variety of problems. A classic example of a stochastic optimal control problem class is portfolio allocation with a consumption rate, which was introduced by Merton (1971). Many authors have attacked the research direction adopted for production planning in manufacturing systems. The literature on the subject includes, for example, the first contribution in the field that derived the control strategies of a quadratic model (Holt et al., 1960) using the calculus of variation principle. Bensoussan et al. (1974) presented some applications of the optimal control theory using the maximum principle to management science and economics, as well as the discrete and continuous stochastic optimal control versions addressed by Sethi et al. (1981, 2000).

Tzafestas et al. (1997) introduced and clarified the concept of model-based predictive control (MBPC) for integrated production planning problems in a stochastic environment. Dobos (2003) investigated the deterministic optimal production and inventory of the HMMS (Holt, Modigliani, Muth and Simon) model. This is considered as a generalization of the Holt et al. (1960) model, which was analysed by El-Gohary et al. (2009), using Pontryagin's maximum principle to determine the optimal control policies. El-Gohary et al. (2007) studied the problem of stochastic production and inventory planning using stochastic optimal control. Based on such studies, we are especially interested in the modelling of a production planning problem in a stochastic environment using the stochastic Pontryagin's maximum principle.

The main objective of this study is to employ a two-level hierarchical control model (as in Kenné and Boukas, 2003), using the stochastic optimal control principle, to optimally to solve the following two problems:

- In the first level of the proposed hierarchy, the desired inventory threshold (the safety stock level that the company wants to keep on hand) and the desired production rate (the most efficient rate desired by the firm) are determined.
- In the second level the proposed hierarchy, the goal will be to keep the inventory level as close as possible to the desired inventory threshold and the rate of production as close as possible to the desired production rate.

This objective will allow the minimization of the total average quadratic cost and the sum of mean square deviations of inventory and production for the first and the second levels, respectively.

In the present research, to solve the stochastic production planning problem in continuous time, we propose a two-level hierarchical control approach. We assume that the product demand rate is stochastic over time, and at both levels, it is assumed that the values of the control variables are restricted; moreover, there are no restrictions on the state variables. The production rate is thus defined as varying in a given control region and the inventory level is allowed to be positive (inventory) or negative (backlogged demands). By applying Pontryagin's maximum principle (Pontryagin et al., 1962) and using the quadratic objective function of Holt et al. (1960), we can formulate the boundary value problems in the first and second levels. We may recall that the first level is based on the formulation of a stochastic linear-quadratic model (SLQM) to find the optimal desired production rate and inventory level over the planning horizon, that minimize the total average quadratic inventory and production cost.

The second level is devoted to a recursive optimization control problem (predictive control) to find the optimal production rate needed to bring the inventory level from the initial point to one which satisfies the demand rate, but at the same time, minimizes the sum of mean square deviations of inventory and production. The solutions are obtained numerically using the Euler-Maruyama scheme.

This paper is organized as follows. In section 2, we define the assumptions used in the model and provide the problem statement. Based on the stochastic Pontryagin's maximum principle, we present the solution of the hierarchical control model in section 3. A numerical example is given in section 4. Sensitivity and results analyses are provided in section 5, and finally, section 6 closes the paper with some concluding remarks.

2. Production control model

2.1. Notations and assumptions

The following notations are used throughout the paper:

T : length of the planning horizon, ($T > 0$),

x^0 : desired initial inventory level,

x_0 : initial inventory level,

$x(t)$: inventory level at time t,

$\bar{x}(t)$: desired inventory level at time t,

$\bar{u}(t)$: desired production rate at time t, ($\bar{u}(t) \geq 0$),

d(t) : demand rate at time t,

c : penalty factor of production, ($c > 0$),

h : penalty factor of inventory ($h > 0$),

c_m : deviation factor for the production rate to deviate from its desired rate $\bar{u}$, ($c_m > 0$),

c_h : deviation factor for the inventory level to deviate from its desired level $\bar{x}$, ($c_h > 0$),

$u(t)$: production rate at time t, ($u(t) \geq 0$),

U_{max}: maximum production rate,

σ_D : standard deviation of demand rate,

w : standard Wiener process,

g : diffusion coefficient.

In this model, we assume that:

(1) The production rate is to be non-negative.
(2) The inventory level is allowed to be negative (backlogged demands).
(3) The planning horizon is finite in continuous time.
(4) The demand rate is assumed to be described by the following time-dependent model (Kiesmüller, 2003):

$$d(t) = \mu_D(t) + \sigma_D \varepsilon(t),$$

where the parameter $\mu_D(t)$ is a deterministic known parameter, and σ_D is also a known non-negative parameter. It is assumed that $\varepsilon(t)$ is the white noise, a normally distributed random variable in time t with zero mean and unity variance: $\varepsilon(t) \sim N(0, 1)$ and $E[\varepsilon(t_1)\varepsilon(t_2)] = 0$ for $t_1 \neq t_2$. We can therefore write the demand rate as a normally distributed function: $d(t) \sim N(\mu_D(t), {\sigma_D}^2)$.

2.2. Problem statement

In this section, we describe a stochastic hierarchical model. The system considered is composed of a single reliable machine producing one type of product, with the demand rate capable of being fulfilled with time. The situation is illustrated in Figure 1.

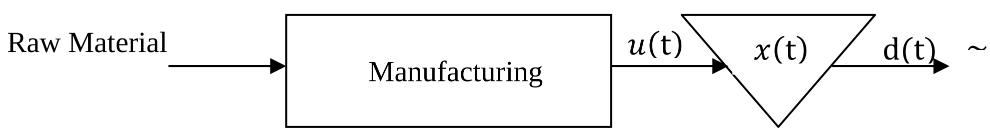


Figure 1. Structure of the system

First, we will consider the following stochastic linear-quadratic optimal control problem (denoted (P_0)) of the stochastic desired production planning problem, with one state variable (desired inventory level) and with one control variable (desired production rate). The objective of the firm is to find the desired optimal production rate needed to bring the inventory level from the initial point to a point satisfying the demand rate which minimizes the total average quadratic inventory and production cost.

The objective function to minimize (see Appendix A) is given by:

$$\text{minimize} \quad J_0(x^0, u) = E\left\{\int_0^T \phi_0(x, u, t)dt + K_0(x(T), T)\right\}, \tag{1}$$

where $\phi_0(x, u, t) = \frac{1}{2}hx(t)^2 + \frac{1}{2}cu(t)^2,$ (2)

subject to:

the stochastic state equation of the inventory level $x(\mathrm{t})$,

$$\dot{x} = f(x, u, w, t) = f(x, u, t) + g\dot{w}, \tag{3}$$

and initial condition $x(0) = x^0$, where $t \in \mathbb{R}, u \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$

and the non-negativity constraint:

$$u(\mathrm{t}) \geq 0, \quad \text{for all } \mathrm{t} \in [0, \mathrm{T}]. \tag{4}$$

Secondly, the initial stochastic production planning problem can be represented as a stochastic optimal control problem denoted (P) with one state variable (inventory level) and with one control variable (production rate), which is to find the optimal production rate satisfying the demand rate over the planning horizon and minimizing the sum of the inventory and production costs. This objective function can be interpreted as meaning that penalty costs are incurred when the inventory level and the production rate deviate from their desired goals (Holt et al., 1960). The function was recently generalized by Dobos (2003).

The objective function for the minimization (see Appendix A) is given by:

$$\text{minimize} \qquad J(x_0, u) = E\left\{\int_0^T \phi(x, u, t)dt + K(x(T), T)\right\}, \tag{5}$$

$$\text{where} \quad \phi(x, u, t) = \frac{1}{2}\mathrm{c_h}[x(\mathrm{t}) - \bar{x}(\mathrm{t})]^2 + \frac{1}{2}\mathrm{c_m}[u(\mathrm{t}) - \bar{u}(\mathrm{t})]^2, \tag{6}$$

subject to:

the stochastic state equation of the inventory level $x(\mathrm{t})$:

$$\dot{x} = f(x, u, w, t) = f(x, u, t) + g\dot{w}, \tag{7}$$

and initial condition $x(0) = x_0$, where $t \in \mathbb{R}, u \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$.

and the non-negativity constraint:

$$u(\mathrm{t}) \geq 0, \quad \text{for all } \mathrm{t} \in [0, \mathrm{T}]. \tag{8}$$

The optimal control to find on $[0, T]$ is u^* to minimize the criterion J:

$$\mathrm{u}^* = \min_{\mathrm{u}[0,\mathrm{T}]} \mathrm{J}(\mathrm{x}_0, \mathrm{u}). \tag{9}$$

In the next section, we will use the above assumptions and Pontryagin's maximum principle to describe our approach to solving the optimal control problems.

3. Solution of the optimal control problems

The hierarchical structure of the proposed control approach is illustrated in Figure 2. Both control problems can be solved by applying the stochastic optimal control principle based on Pontryagin's maximum principle using the Euler-Maruyama scheme.

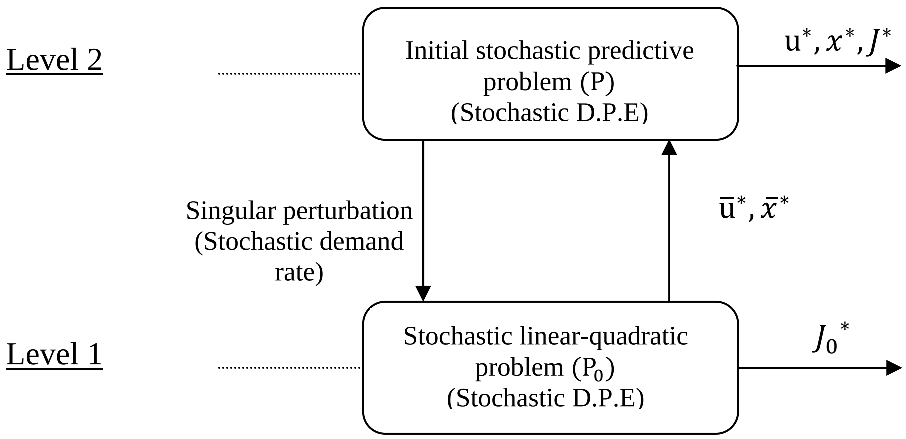


Figure 2. Hierarchical control approach

3.1. Solution of the optimal control problem $(\mathbf{P_0})$

To find the solution of the stochastic linear-quadratic optimal control problem defined by (1) and (3), we will assume that $V(x,t)$, known as the value function related to the objective function (1) from t to T, is given by:

$$\mathrm{V}(x,\mathrm{t}) = \min_u \mathrm{E}\left\{\int_{\mathrm{t}}^{\mathrm{T}} \phi_0(x,u,\mathrm{t})\mathrm{dt} + K_0(x(T),T)\right\}. \tag{10}$$

Let $t \in [0,T]$. The optimality principle of Bellman holds that the optimal trajectory over the interval $[0,T]$ contains the optimal trajectory over the interval $[t,T]$ with initial condition $x_1 = x(t)$. In other words:

$$\mathrm{V}(x,\mathrm{t}) = \min_u \mathrm{E}\{\phi_0(x,u,\mathrm{t})\mathrm{dt} + \mathrm{V}(x+\mathrm{d}x,\mathrm{t}+\mathrm{dt})\}. \tag{11}$$

Applying the Taylor expansion:

$$V(x+dx,t+dt)=V(x,t)+V_t dt+V_x dx+\frac{1}{2}V_{tt}(dt)^2+\frac{1}{2}V_{xx}(dx)^2+V_{xt}dxdt+\ldots, \quad (12)$$

after simplification, we neglect the higher-order terms (Davis, 1977) such as $(dt)^2$, $dwdt$ and replacing $(dw)^2$ with (dt), we obtain the Itô formula

$$\mathrm{dV}(x,\mathrm{t})=(\mathrm{V_t}+\mathrm{V}_x\mathrm{f}+\frac{1}{2}\mathrm{g}^2\mathrm{V}_{xx})\mathrm{dt}+\mathrm{V}_x\mathrm{gdw}, \quad (13)$$

and:

$$\mathrm{V}(x,\mathrm{t})=\min_{\mathrm{u}}\mathrm{E}\left\{\phi_0\mathrm{dt}+\mathrm{V}(x,\mathrm{t})+\mathrm{V_t dt}+\mathrm{V}_x\mathrm{fdt}+\frac{1}{2}\mathrm{g}^2\mathrm{V}_{xx}\mathrm{dt}+\mathrm{V}_x\mathrm{gdw}+0(\mathrm{dt})\right\}. \quad (14)$$

Given that the variables dw and x are independent, we can write:

$$\mathrm{EV_x gdw}=\mathrm{EV_x gEdw}=0, \quad (15)$$

Hence, we can have the Hamiltonian-Jacobi-Bellman equation satisfied by the value function

$$0=\min_u\left\{\phi_0+\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}f+\frac{1}{2}g^2\frac{\partial^2 V}{\partial x^2}\right\}, \quad (16)$$

with boundary condition: $V(x,T)=K_0(x(T),T)$. (17)

To solve the nonlinear partial differential equation, we can approximate the value function by a quadratic polynomial and find the optimal coefficient of this polynomial. The proposed method is based on Pontryagin's maximum principle and the derivatives of the value function are obtained by the following adjoint equations for the stochastic optimal control problem:

$$p(t)=\frac{\partial V(x,t)}{\partial x}=V_x(x,t)\,; \quad p_x(t)=\frac{\partial p(t)}{\partial x}=V_{xx}(x,t) \quad (18)$$

The stochastic version of the optimality conditions can be obtained using the Hamiltonian function:

$$\begin{aligned}-\mathrm{V_t}&=\min_u\left\{\phi_0(x,u,\mathrm{t})+\mathrm{V}_x\mathrm{f}+\frac{1}{2}\mathrm{g}^2\mathrm{V}_{xx}\right\}\\&=\min_u H(x,u,p,p_x,t).\end{aligned} \quad (19)$$

We note that the Hamiltonian function in the deterministic version is:

$$\mathrm{H(x,u,p,t)}=\phi_0(\mathrm{x,u,t})+\mathrm{V_x f}, \quad (20)$$

and in the stochastic version is:

. $$H(x,u,p,p_x,t)=H(x,u,p,t)+\frac{1}{2}g^2V_{xx}, \quad (21)$$

with H: $[0,T] \times \mathbb{R}^n \times u \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$

$$H(x,u,p,p_x,t) = H(x,u,p,t) + \frac{1}{2} tr\{p_x g g^T\}$$

$$= \phi_0(x,u,t) + f^T(x,u,t)p + \tfrac{1}{2} tr\{p_x g(x,u,t) g^T(x,u,t)\}. \qquad (22)$$

Assuming that we have an optimal control $u^*(x,p,p_x,t)$ which solves the stochastic optimal control problem:

$$H^*(x,p,p_x,t) = H(x,u^*(x,p,p_x,t),p,p_x,t) \qquad (23)$$

$$= \phi_0(x,p,p_x,t) + f^T(x,p,p_x,t)p$$

$$+ \tfrac{1}{2} tr\{p_x g(x,p,p_x,t) g^T(x,p,p_x,t)\} \qquad (24)$$

$$= -V_t, \qquad (25)$$

we can write the stochastic differential equation for the state

$$\mathrm{d}x = \mathrm{f}(x,u^*,\mathrm{t})\mathrm{dt} + \mathrm{g}(x,u^*,\mathrm{t})\mathrm{dw}$$

$$= \mathrm{H}_\mathrm{p}^*(x,\mathrm{p},\mathrm{p}_x,\mathrm{t})\mathrm{dt} + \mathrm{g}(x,\mathrm{p},\mathrm{p}_x,\mathrm{t})\mathrm{dw}, \qquad (26)$$

and for the adjoint, we use Itô’s lemma (Cohn, 1980) to determine the process followed by p(t)

from (18)

$$\mathrm{dp} = \mathrm{V}_{xt}\mathrm{dt} + \mathrm{V}_{xx}\mathrm{dx} + \tfrac{1}{2}\mathrm{V}_{xxx}(\mathrm{d}x)^2 \qquad (27)$$

$$= \left[\mathrm{V}_{xt} + \mathrm{V}_{xx} f + \tfrac{1}{2} tr\{\mathrm{V}_{xxx} g g^T\}\right] \mathrm{dt} + \mathrm{V}_{xx} g dw. \qquad (28)$$

To obtain V_{xt} we use the partial derivative of V_t with respect to x

$$-\mathrm{V}_{xt}(\mathrm{x},\mathrm{t}) = \mathrm{H}_x^* + \mathrm{H}_\mathrm{p}^* \frac{\partial \mathrm{p}}{\partial x} + \mathrm{H}_{\mathrm{p}_x}^* \frac{\partial \mathrm{p}_x}{\partial x}$$

$$= H_x^* + \mathrm{V}_{xx} f + \tfrac{1}{2} tr\{\mathrm{V}_{xxx} g g^T\}. \qquad (29)$$

Substituting (29) into (28), we get

$$\mathrm{dp} = -H_x^* \mathrm{dt} + \mathrm{V}_{xx} g dw$$

$$= -H_x^* \mathrm{dt} + \mathrm{p}_x g dw. \tag{30}$$

Now, we can give the necessary conditions of optimality for stochastic control in the case of minimization:

$$H(x,u,p,p_x,t) = \phi_0(x,u,t) + f^T(x,u,t)p + \frac{1}{2} tr\{p_x g(x,u,t) g^T(x,u,t)\}$$

$$\begin{cases} \mathrm{d}x^* = H_p^* dt + g dw \\ \mathrm{dp}^* = -H_x^* \mathrm{dt} + \mathrm{p_x} g dw \\ x^*(0) = x^0 \\ \mathrm{p}^*(\mathrm{T}) = K_x(x(T),T) \\ \mathrm{H}(x^*(t), u^*(t), p^*(t), {p_x}^*(t), t) \le H(x^*(t), u(t), p^*(t), {p_x}^*(t), t). \end{cases} \tag{31}$$

Thus the optimal inventory level and adjoint are obtained from $\mathrm{d}x^*$ and dp^* and the optimal production rate is obtained by minimizing the Hamiltonian function.

The Hamiltonian function:

$$H(x,u,p,p_x,t) = \frac{1}{2} c\, u^2(t) + \frac{1}{2} \mathrm{h} x^2(t) + \big(u(t) - \mu_\mathrm{D}(\mathrm{t})\big) p(t) + \frac{1}{2} p_x {\sigma_\mathrm{D}}^2$$

and the two-point boundary value problem of (P_0):

$$(\mathrm{TPBVP}_0) \begin{cases} \mathrm{d}x^* = \big(u(t) - \mu_\mathrm{D}(\mathrm{t})\big) dt - \sigma_\mathrm{D} dw \\ \mathrm{dp}^* = -\mathrm{h} x(t) \mathrm{dt} - \mathrm{p}_x(t) \sigma_\mathrm{D} dw \\ u^*(x,\mathrm{t}) = \frac{\partial H}{\partial u(\mathrm{t})} = \frac{-\mathrm{p(t)}}{\mathrm{c}} \\ x^*(0) = x^0 \, ; \; x^*(\mathrm{T}) = free \\ \mathrm{p}^*(\mathrm{T}) = 0. \end{cases} \tag{32}$$

The stochastic differential equations are solved numerically using the Euler-Maruyama scheme (see Appendix A) of the problem (P_0) obtained by applying Pontryagin's maximum principle, after which the optimal inventory trajectory and the optimal adjoint trajectory are given by the following forms, respectively:

$$\mathrm{Y_{n+1}}^{(1)} = \mathrm{Y_n}^{(1)} + \left(\frac{-\mathrm{Z_n}^{(1)}}{\mathrm{c}} - \mu_\mathrm{Dn}\right) \mathrm{dt} + (-\sigma_\mathrm{D}) \sqrt{\mathrm{dt}}\, \varepsilon_\mathrm{n}, \tag{33}$$

$$\mathrm{Z_{n+1}}^{(1)} = \mathrm{Z_n}^{(1)} - \mathrm{h} \mathrm{Y_n}^{(1)} \left(\mathrm{dt} + \frac{(-\sigma_\mathrm{D})\sqrt{\mathrm{dt}}\,\varepsilon_\mathrm{n}}{\left(\frac{-\mathrm{Z_n}^{(1)}}{\mathrm{c}} - \mu_\mathrm{Dn}\right)} \right) / \mathrm{dt}. \tag{34}$$

The initial value of the adjoint is given using the simple shooting method (Stoer and Bulirsch, 1993).

$$Z_0^{(1)} = \left(\frac{-Y_0^{(1)}}{dt} + \mu_{D0}\right) c, \tag{35}$$

The optimal desired production rate is given by the optimization of the Hamiltonian function (Seierstad and Sydsaeter, 1987), and from (4) and (32), we get:

$$u^*(x, t) = \max\left[0, -\frac{p(t)}{c}\right]. \tag{36}$$

To specify the optimal control for the stochastic linear-quadratic problem, we used the sat function (Sethi and Thompson, 2000):

$$u^*(x, t) = sat\left[0, U_{max}; -\frac{P(t)}{c}\right], \tag{37}$$

$$= \begin{cases} 0 & \text{if } -\frac{p(t)}{c} < 0 \\ -\frac{p(t)}{c} & \text{if } 0 \leq -\frac{p(t)}{c} \leq U_{max} \\ U_{max} & \text{if } -\frac{p(t)}{c} > U_{max}. \end{cases} \tag{38}$$

We then obtained the optimal desired inventory level $\bar{x}^*$and the optimal desired production rate $\bar{u}^*$ by taking the average of the optimal inventory levels and the optimal production rates, respectively.

3.2. Solution of the optimal control problem (P)

To find the solution of the stochastic predictive control problem defined by (5) and (7), we take the same equations used in the first solution of the problem(P_0).

Thus the optimal inventory level and adjoint are obtained from dx^* and dp^* and the optimal production rate is obtained by minimizing the Hamiltonian function. Such a function is given by:

$$H(x,u,p,p_x,t) = \frac{1}{2}c_h\big(x(t)-\bar{x}(t)\big)^2 + \frac{1}{2}c_m\big(u(t)-\bar{u}(t)\big)^2 + \big(u(t)-\mu_D(t)\big)p(t) + \frac{1}{2}p_x{\sigma_D}^2$$

and the two-point boundary value problem of (P):

$$(\text{TPBVP})\begin{cases} dx^* = \big(u(t)-\mu_D(t)\big)dt - \sigma_D dw \\ dp^* = -c_h(x(t)-\bar{x}(t))dt - p_x(t)\sigma_D dw \\ u^*(x,t) = \frac{\partial H}{\partial u(t)} = -\frac{p(t)}{c_m} + \bar{u}(t) \\ x^*(0) = x_0\ ;\ x^*(T) = free \\ p^*(T) = 0. \end{cases} \tag{39}$$

The stochastic differential equations of the problem (P) are solved numerically using the Euler-Maruyama scheme (see Appendix A) to find the optimal inventory trajectory and the optimal adjoint trajectory given by the following forms, respectively:

$$Y_{n+1}{}^{(2)} = Y_n{}^{(2)} + \left(\frac{-Z_n{}^{(2)}}{c_m} + \bar{u}_n - \mu_{Dn}\right)dt + (-\sigma_D)\sqrt{dt}\varepsilon_n, \tag{40}$$

$$Z_{n+1}{}^{(2)} = -Z_n{}^{(2)} + c_h(Y_n{}^{(2)} - \bar{x}_n)\left(dt + \frac{(-\sigma_D)\sqrt{dt}\varepsilon_n}{\left(\frac{-Z_n{}^{(2)}}{c_m} + \bar{u}(t) - \mu_{Dn}\right)}\right)/dt.$$

(41)

The initial value of the adjoint is given using the simple shooting method (Stoer and Bulirsch, 1993).

$$Z_0{}^{(2)} = \left(\frac{-Y_0{}^{(2)}}{dt} - \bar{u}_0 + \mu_{D0}\right)c_m, \tag{42}$$

The optimal production rate is given by:

$$u^*(x,t) = \max\left[0, -\frac{p(t)}{c_m} + \bar{u}(t)\right]. \tag{43}$$

To specify the optimal control for the stochastic predictive problem, we also used the *sat* function (Sethi and Thompson, 2000), to obtain:

$$u^*(x,t) = sat\left[0, U_{max}; -\frac{p(t)}{c_m} + \bar{u}(t)\right], \tag{44}$$

$$= \begin{cases} 0 & \text{if} \quad -\frac{p(t)}{c_m} + \bar{u}(t) < 0 \\ -\frac{p(t)}{c_m} + \bar{u}(t) & \text{if} \quad 0 \leq -\frac{p(t)}{c_m} + \bar{u}(t) \leq U_{max} \\ U_{max} & \text{if} \quad -\frac{p(t)}{c_m} + \bar{u}(t) > U_{max}. \end{cases} \qquad (45)$$

We will apply the proposed approach to solve a numerical example of the stochastic production planning problem.

4. Numerical example

In this section, we present a numerical example to illustrate the results obtained. It is assumed that the demand rate is a normally distributed function of time, defined as: $d(t) = \mu_D(t) + \sigma_D N(0,1)$ and other parameters are presented in Table 1:

Table 1: Data of numerical example

Parameter	T	c	h	c_m	c_h	x^0	x_0	U_{max}	μ_D	σ_D
Value	120	20	80	12	3	0.8147	0.8147	10.5	10	4.5

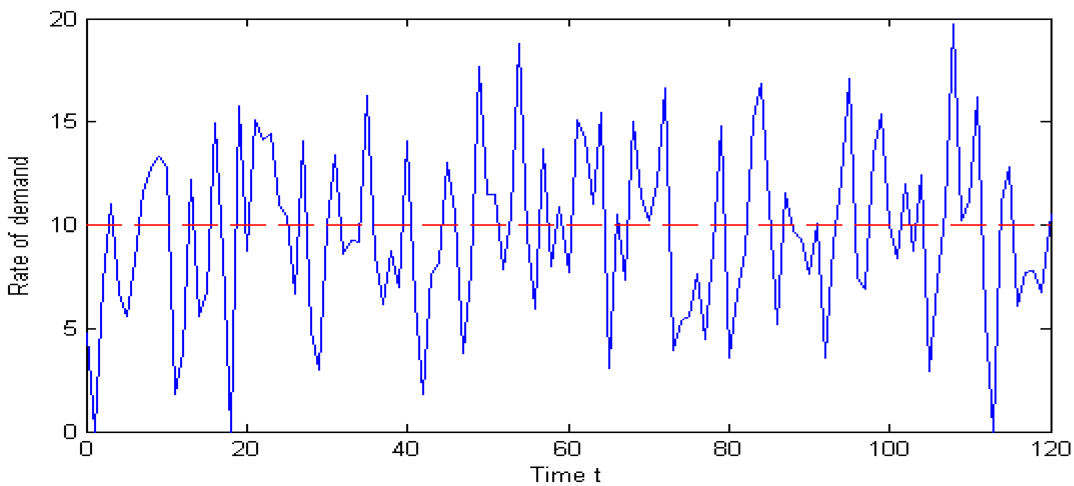


Figure 3. Demand rate against time

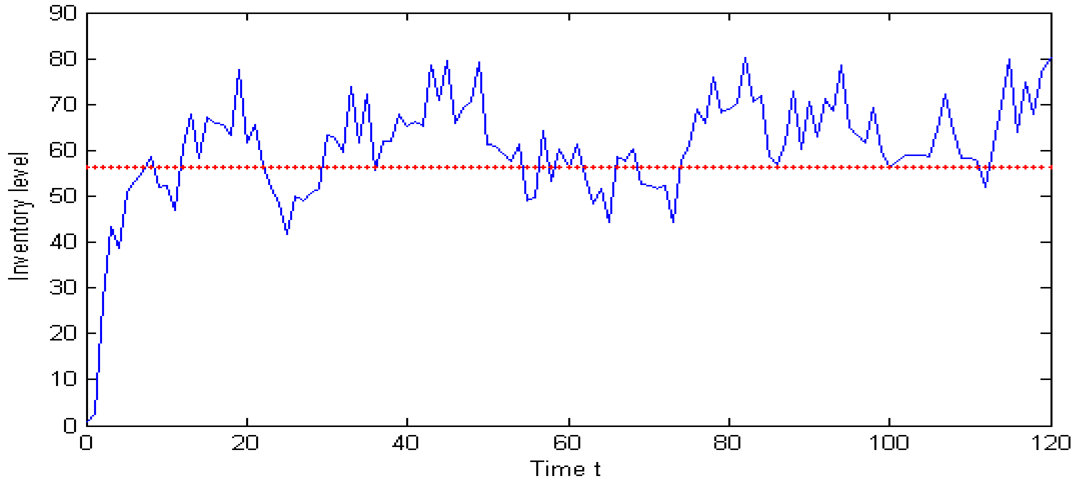


Figure 4. Optimal inventory levels against time

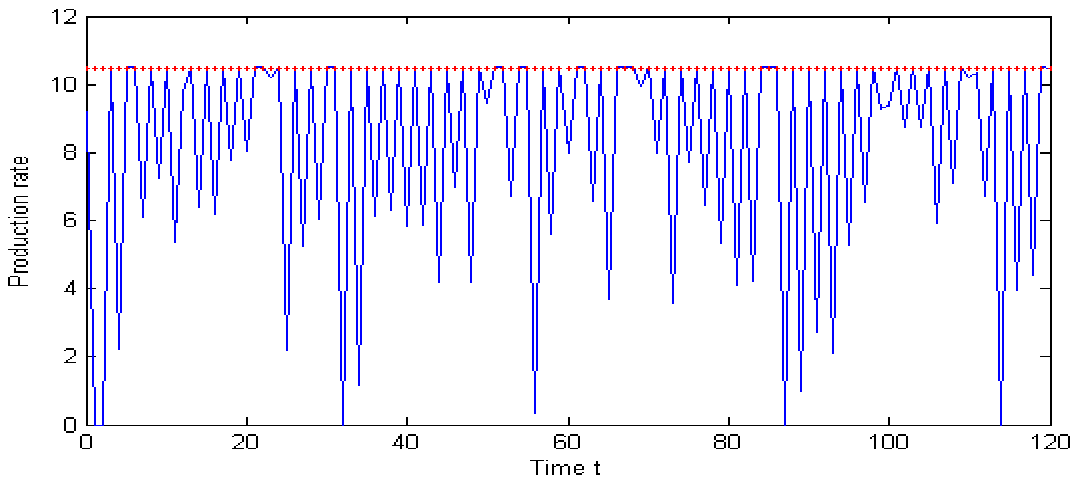


Figure 5. Optimal production rates against time

The simulation results are given in Figures 3, 4 and 5, and show that the optimal inventory level x^* increases with time to converge to its optimal desired inventory level $\bar{x}^*$. The optimal production rate u^* must satisfy the normally distributed demand rate until it reaches its optimal desired production rate $\bar{u}^*$ at the end of the planning horizon period. For our numerical example, we obtained an optimal desired production rate with a value $\bar{u}^* = 10.4$ and an optimal desired inventory level with a value $\bar{x}^* = 56.28$, which represent the production rate and inventory threshold desired by the firm to respond to random demand with $\mathrm{d(t)} \sim \mathrm{N}(10, 4.5)$.

The next section presents the validation of the proposed approach through a sensitivity analysis and illustration of its usefulness while examining how the results obtained vary with changes in some monetary and non-monetary parameters values.

5. Sensitivity analysis

From the figures and tables presented in this section, we notice two parameters that influence the results obtained. Firstly, we have the so-called monetary parameters that include c, h, c_m and c_h and the non-monetary parameter, σ_D. These parameters have an influence on the optimal desired inventory level $\bar{x}^*$, the optimal desired production rate $\bar{u}^*$, the optimal inventory level x^*, the optimal production rate u^*, the total average quadratic cost $J_0{}^*$ and the total average deviation cost J^* of inventory and production.

In the interpretations that follow, the simulation results show that the optimal desired production rate remains constant at $\bar{u}^* = 10.4$. This allows regular production around the customer's demand using the finite products stock as a buffer to satisfy fluctuating demand.

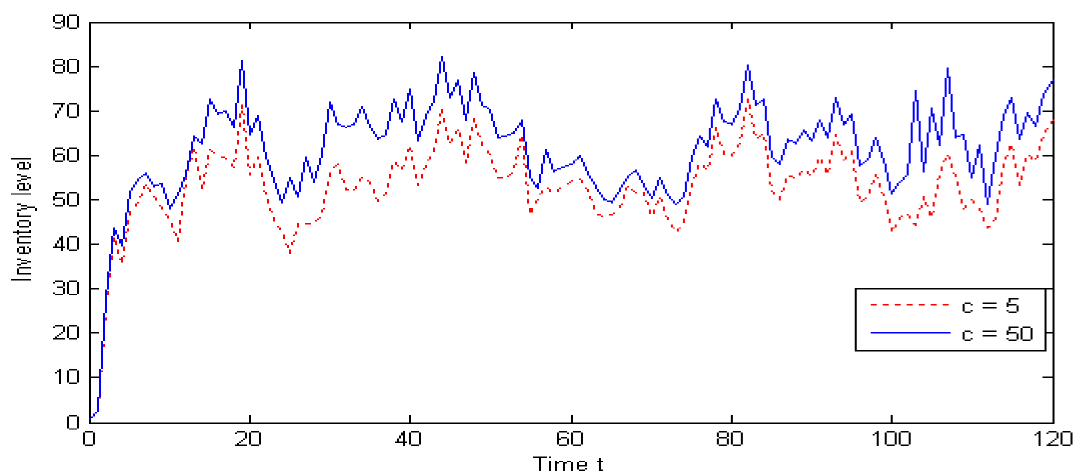
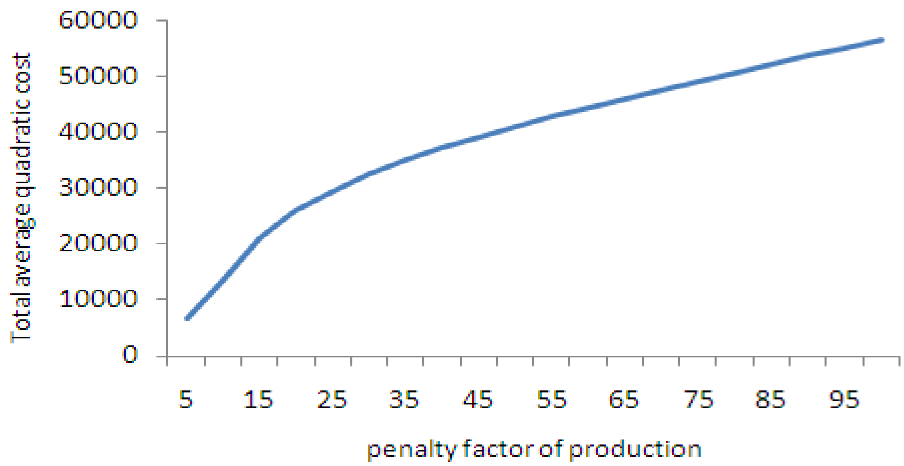


Figure 6. Optimal inventory level against time for c = 5, 50.

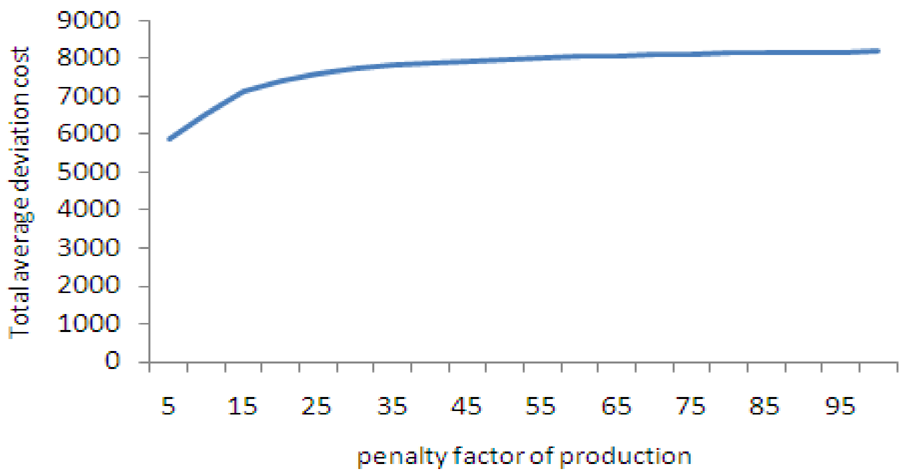
Table 2: Effect of penalty factor c

c	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**
$\bar{x}^*$	50.5	53.1	55.2	56.3	56.9	57.3	57.6	57.9	58	58.2

c	**55**	**60**	**65**	**70**	**75**	**80**	**85**	**90**	**95**	**100**
$\bar{x}^*$	58.3	58.4	58.48	58.55	58.61	58.66	58.71	58.75	58.79	58.82



(a)



(b)

Figure 7. Total average cost against penalty factor of production

From Figures 6, 7 and Table 2, we note that when the penalty factor c increases, the optimal desired inventory level $\bar{x}^*$ increases as well, as does the optimal inventory level x^*. This result is logical because the more we penalize the production, the faster we produce in order to respond to random demand, leading to a stock surplus. This causes an increase in the total average quadratic cost J_0^* and in the total average deviation cost J^*.

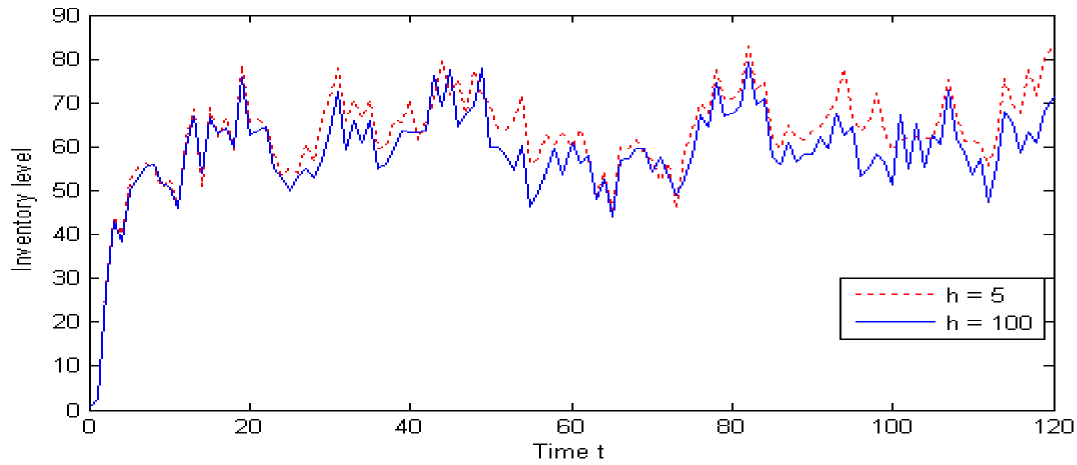
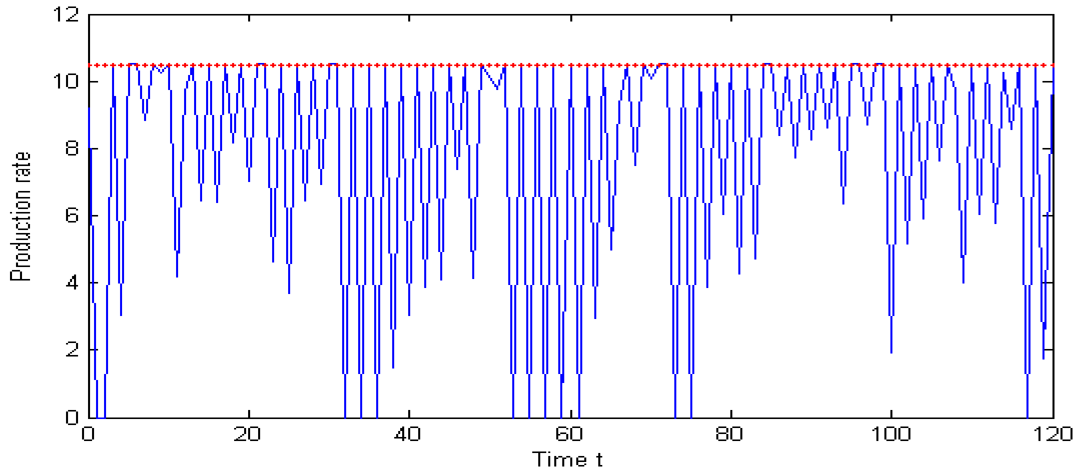


Figure 8. Optimal inventory level against time for h $= 5, 100$

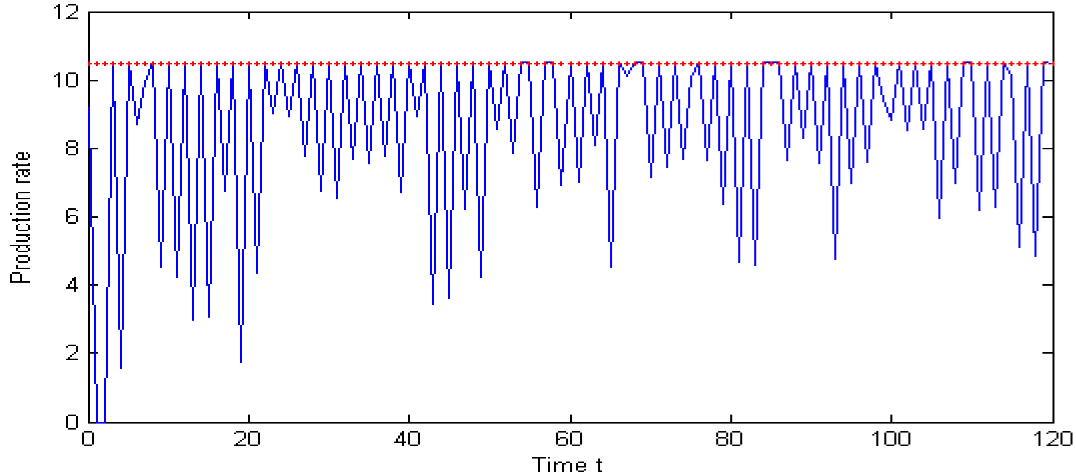
Table 3: Effect of penalty factor h

h	5	10	15	20	25	30	35	40	45	50
$\bar{x}^*$	59.26	59.06	58.86	58.66	58.46	58.26	58.06	57.87	57.67	57.47
J_0^*	7204.6	9138.1	10964	12684	14301	15817	17234	18556	19783	20917
J^*	8323.4	8261.1	8199.1	8137.4	8076	8014.9	7954.1	7893.7	7833.5	7773.5
h	**55**	**60**	**65**	**70**	**75**	**80**	**85**	**90**	**95**	**100**
$\bar{x}^*$	57.27	57.07	56.87	56.67	56.48	56.28	56.08	55.88	55.68	55.48
J_0^*	21962	22921	23793	24583	25292	25923	26479	26961	27371	27710
J^*	7713.9	7654.7	7595.6	7536.9	7478.5	7420.5	7362.7	7305.4	7248.3	7191.3

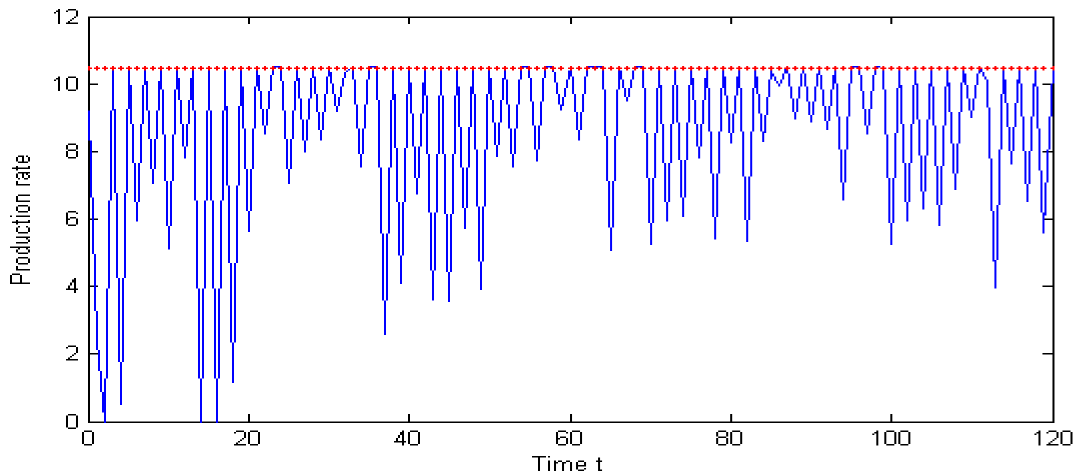
From Figure 8 and Table 3, we notice that when the penalty factor h increases, the optimal desired inventory level $\bar{x}^*$ decreases; the optimal inventory level x^* decreases as well because more the inventory is penalized, the slower we produce in order to satisfy random demand, which in turn causes a decrease in products stocks. The optimal inventory level x^* decreases accordingly. Regarding the total average quadratic cost $J_0{}^*$, it will increase, while the total average deviation cost J^* decreases.



(a) $c_m = 10$

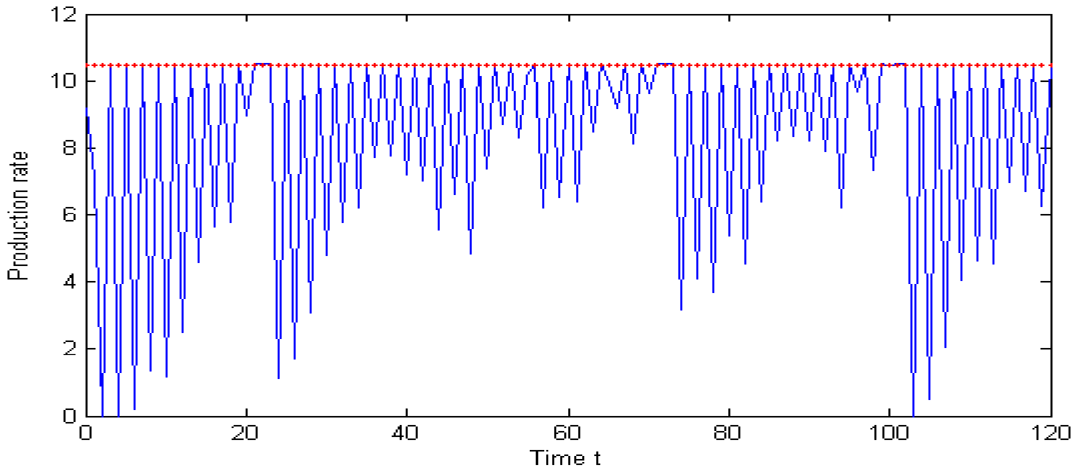


(b) $c_m = 14$

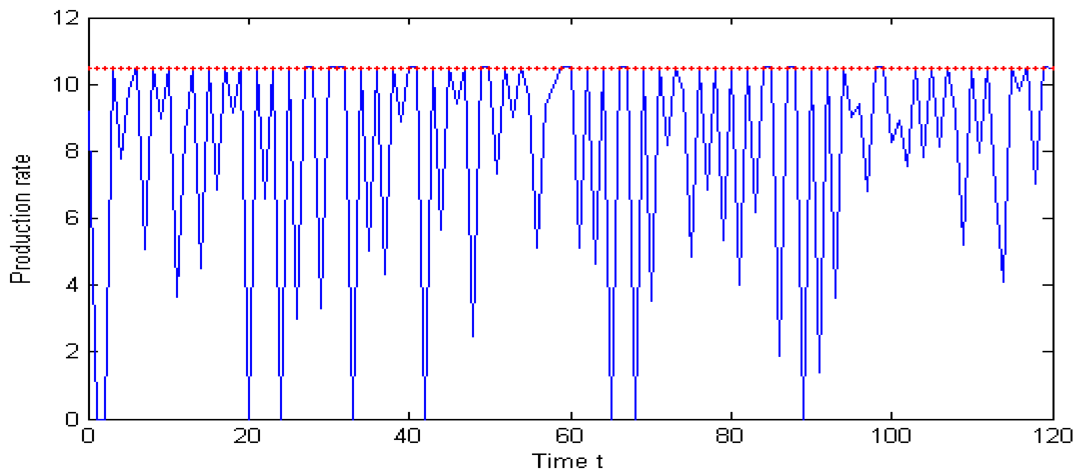


(c) $c_m = 18$

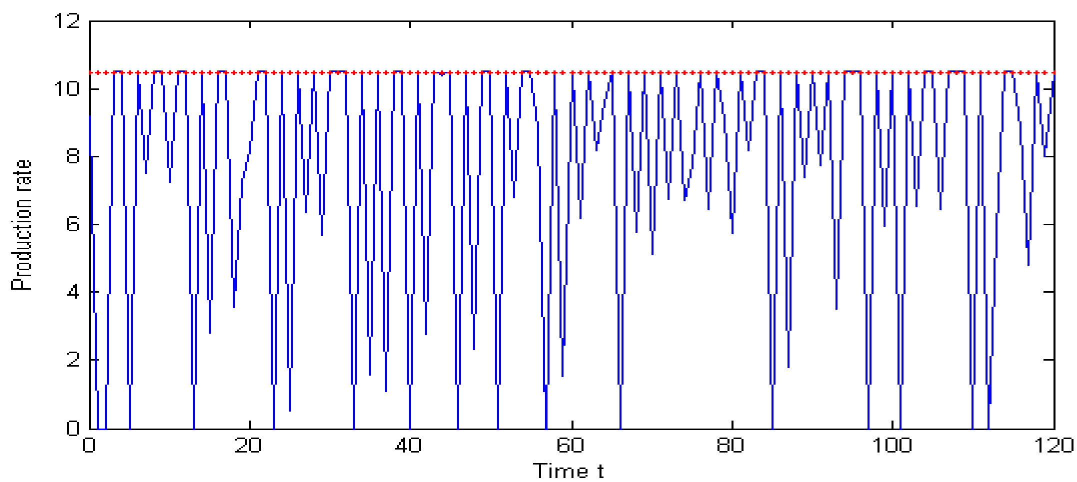
Figure 9. Optimal production rate against time for $c_m = 10, 14, 18$



(a) $c_h = 1$



(b) $c_h = 6$



(c) $c_h = 9$

Figure 10. Optimal production rate against time for $c_h = 1, 6, 9$

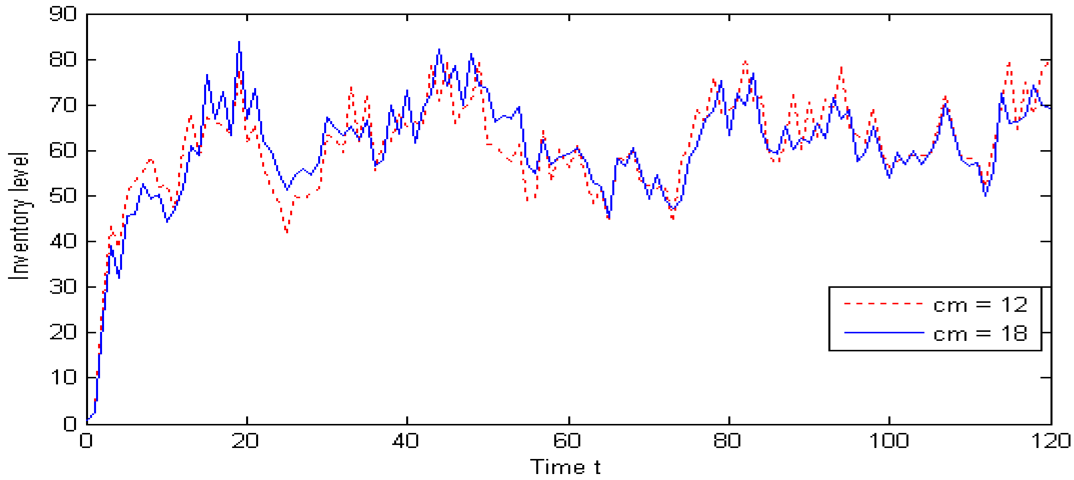


Figure 11. Optimal inventory level against time for $c_m = 12, 18$

Table 4: Effect of deviation factor c_m

c_m	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
J^*	7017.4	7220.9	7420.5	7613.1	7797.3	7844.3	7860.6	7878.2	7896.3	7914	7931.1

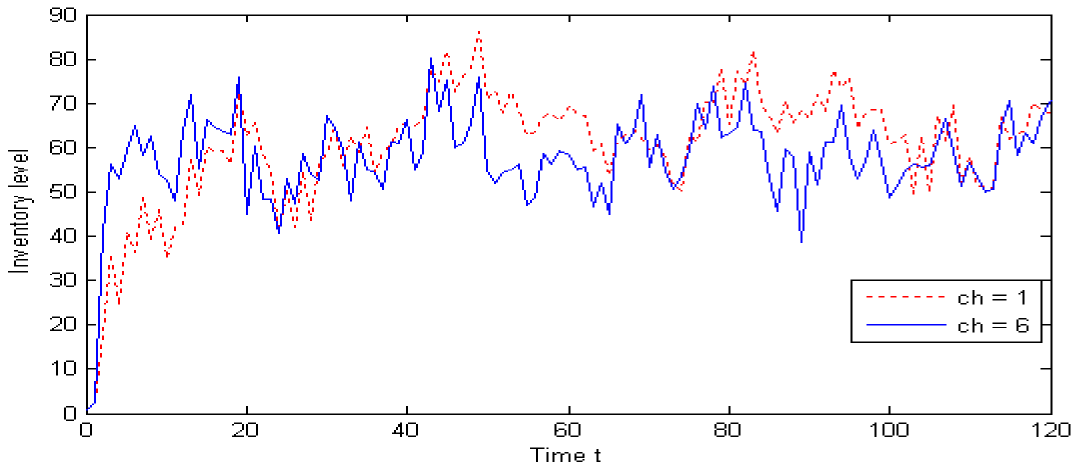


Figure 12. Optimal inventory level against time for $c_h = 1, 6$

Table 5: Effect of deviation factor c_h

c_h	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
J^*	2681	5264.2	7420.5	9086.6	10813	12699	14866	17450	21449	24825

From Figures 9, 10, 11, 12 and Tables 4 and 5, we can see that when the deviation factor of production c_m increases, it causes an increase in production (see Figures 9(a), 9(b), 9(c)). Because of faster production, we have more products in stock (see Figure 11). This leads to a high deviation between the optimal inventory level x^* and its goal. We also note that the more the deviation factor of the inventory c_h increases, the more the inventory level moves away from its goal because the more the inventory is penalized, the slower we produce as well (see Figures 10(a), 10(b), 10(c)), and stock fewer products (see Figure 12), and the total average deviation cost J^* increases in both cases (see Tables 4, 5).

In the first level, there is no change, and the optimal desired inventory level remains constant at $\bar{x}^* = 56.28$ because it is independent of c_m and c_h. Therefore, the total average quadratic cost remains constant at $J_0{}^* = 25923$.

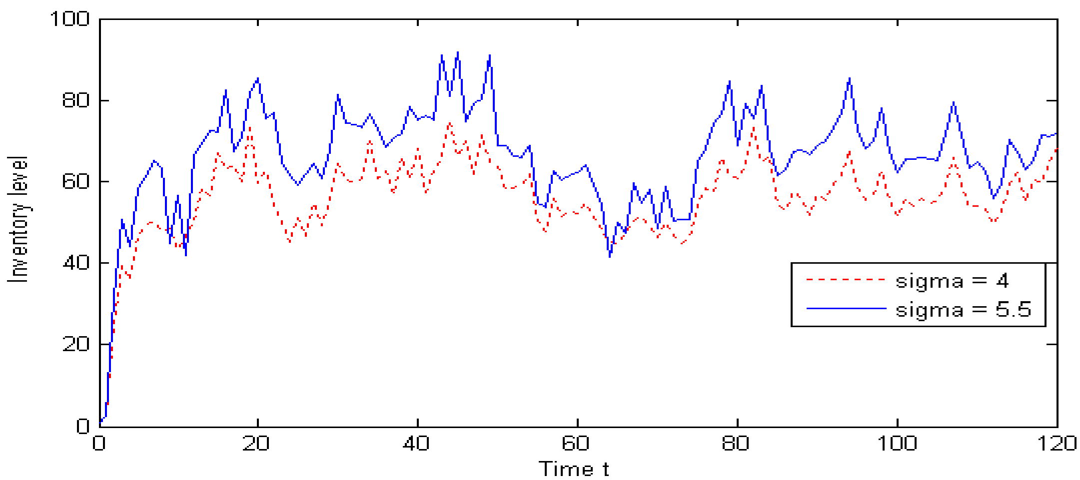


Figure 13. Optimal inventory level against time for $\sigma_D = 4, 5.5$

Table 6: Effect of standard deviation σ_D

σ_D	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5
$\bar{x}^*$	31.61	35.13	38.59	42.29	45.70	49.21	52.75	56.28	59.8	63.32
$J_0{}^*$	4572.7	5107.3	6444.7	8771.3	11742	15605	20358	25923	32328	39570
J^*	3176.8	3656.5	4361.5	4989.6	5320.4	6031.7	6704.4	7420.5	8186.5	9003.5

From Figure 13 and Table 6, when the standard deviation factor σ_D increases, the optimal desired inventory level $\bar{x}^*$ increases because the increase in the variability of the demand rate requires to have a higher level, the cumulative demand decreases. That is why the total average quadratic cost $J_0{}^*$ and the total average deviation cost J^* increase.

6. Conclusions

In this study, we have presented a hierarchical optimal control policy for a manufacturing system, composed of one machine producing one product type, with stochastic demand. In the first level, we have shown how to determine the optimal desired inventory level and production rate based on a stochastic linear-quadratic control problem. In the second level, we have determined the optimal production strategy based on a stochastic predictive control to maintain the production rate and the inventory level as close as possible to their goals. The Pontryagin's maximum principle was applied both in level 1 and level 2. A numerical example is presented, and the results obtained show that the hierarchical control policy has minimized the total average quadratic cost and the sum of the mean square deviations of inventory and production for the first and the second levels, respectively. The extension of this work for reverse logistics systems with stochastic demand is identified as a topic of future research.

Appendix A

A.1. The sum of the mean square deviations of inventory and production to minimize can be written in this form:

$$J^* = \text{Min}\,\text{E}\{\text{J}\} = \text{E}\left\{\int_0^{\text{T}}\left[\frac{1}{2}\text{c}_\text{h}.\,(x^*(\text{t}) - \bar{x}^*(t))^2 + \frac{1}{2}\text{c}_\text{m}.\,(u^*(\text{t}) - \bar{u}^*(\text{t}))^2\right].\,\text{dt}\right\},$$

and the stochastic state equation of the inventory level $x(\text{t})$:

$$\dot{x}(t) = u(t) - \text{d}(\text{t}),$$

We can numerically simulate J^* using the Euler-Maruyama scheme. We approach:

$$x_{t+dt} = x_t + \int_t^{t+dt} u(\text{s})\,ds - \int_t^{t+dt} d(\text{s})\,ds$$

$$= x_t + \int_t^{t+dt} u(\text{s})\,ds - \int_t^{t+dt} (\mu_\text{D}(\text{s}) + \sigma_\text{D}\varepsilon(\text{s}))\,ds$$

by

$x_{t+dt} \approx x_t + (u_t - \mu_{Dt})dt + (-\sigma_D)\sqrt{dt}\varepsilon_t$

we obtain:

$$y_{n+1} = y_n + (u_n - \mu_{Dn})dt + (-\sigma_D)\sqrt{dt}\varepsilon_n$$

we define the inventory position at the beginning of period n after the decision (Kiesmüller, 2003) as:

$$y_s = y_n + u_n dt$$

Here, the Euler-Maruyama scheme assumes that the integrands are constants on the integration interval since the random variable d_n is normally and independent of variance dt: We can write y_{n+1} as:

$y_{n+1} = y_s - (\mu_{Dn} dt + \sigma_D\sqrt{dt}\varepsilon_n)$

$$J^* = \int_0^T \left[\frac{1}{2} c_h . \left(y_s - (\mu_{Dn} dt + \sigma_D\sqrt{dt}\varepsilon_n) - \bar{x}^*(t)\right)^2 + \frac{1}{2} c_m . (u^*(t) - \bar{u}^*(t))^2\right] . dt$$

$$J^* = \int_0^T \frac{1}{2} c_h . \left(y_s(t) - \bar{x}^*(t)\right)^2 dt + \int_0^T \frac{1}{2} c_m . \left(u(t) - \bar{u}^*(t)\right)^2 . dt$$
$$+ \sum_{j=1}^{N} (-c_h)(y_s(j) - \bar{x}_j{}^*)\,(\mu_{Dj} dt + \sigma_D dw(j))$$
$$+ \sum_{j=1}^{N} \left(\frac{1}{2} c_h\right) (\mu_{Dj} dt + \sigma_D dw(j))^2$$

A.2. The total average quadratic inventory and production cost to minimize can be written in the same form as the last one:

$$J_0{}^* = \int_0^T \frac{1}{2} h . \left(y_s(t)\right)^2 dt + \int_0^T \frac{1}{2} c . \left(u(t)\right)^2 . dt + \sum_{j=1}^{N} (-h) y_s(j)\,(\mu_{Dj} dt + \sigma_D dw(j))$$
$$+ \sum_{j=1}^{N} \left(\frac{1}{2} h\right) (\mu_{Dj} dt + \sigma_D dw(j))^2$$

where dt: timestep.

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