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Optimal Safety Stocks and Preventive Maintenance Periods in Unreliable Manufacturing Systems.

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Abstract
We consider a manufacturing system with preventive maintenance that produces a single part type. An inventory is maintained according to a machine age dependent hedging point policy. We conjecture that, for such a system, the failure frequencies can be reduced through preventive maintenance resulting in possible increase in system performance. Traditional preventive maintenance policies, such as age replacement, periodic replacement, are usually studied without finished goods inventories. In the cases where the finished goods inventories are considered, restrictive assumptions are used, such as not allowing breakdown during the stock build up period and during backlog situations due to the complexity of the mathematical model. In order to solve this problem, we develop a more realistic mathematical model of the system, and derive expressions of the overall incurred cost used as the basis for optimal determination of the jointly production and preventive maintenance policies (i.e. production rates and preventive maintenance frequency, depending on inventory levels of the produced parts). Such a cost consists of inventory, backlog, corrective and preventive maintenance costs. The work reported here has a significant practical application (no restriction on failures occurrence and backlog situations) in the context of production planning of manufacturing systems. Numerical examples are included to illustrate the importance and the effectiveness of the proposed methodology.

Keywords: Preventive maintenance, Inventory, Production, Reliability, Manufacturing systems.

1. Introduction

A failure prone part production inventory system is considered in this paper. The system produces a single product type to satisfy an exogenous demand process. To hedge against the uncertainties in the both production and the demand processes, provision for finished inventory buffer between the system and the demands is kept. Demands that arrive when the inventory buffer is empty are back ordered and are, therefore not lost as in available models (Das and Sarkar (1999), Rezg et al. (2004) and references therein). It has been largely shown in the literature that implementing preventive maintenance strategies for several randomly failing production units can be an effective way to extend their lives and reduce operating costs (Barlow and Proschan (1965), Savsar (1997), Chelbi and Ait-Kadi

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(2004) and references therein). The reader is referred to Savsar (2006) for details on other maintenance policies and their effects on the productivity and availability of a manufacturing system. A overview of relevant literature reveals that significant contributions, in the performances optimisation of manufacturing systems, have been proposed based on (i) preventive maintenance, (ii) production control and (iii) jointly production and maintenance optimisation models. Those models are considered individually or simultaneously and are restricted to simplified assumptions that sometimes provide less realistic preventive maintenance or production policies.

In the last few decades, maintenance planning has been an active area of research focused on the reliability theory as presented recently by Chelbi and Ait-Kadi (2004). Hence, a replacement policy which ensures maximum utilisation of the useful life of a component before its preventive replacement is an obvious option for large and costly components. Age replacement policy (ARP) is one such option over block replacement policy (BRP) or group replacement policy (GRP). For details on such policies, the reader is referred to Barlow and Proschan (1965), Ajodhya and Damodar (2004) and references therein. One of the basic and simple replacement policies is the age replacement policy, where the unit is replaced upon a failure or a prefix age, whichever occurs first (see Hong and Jionghua (2003), Ajodhya and Damodar (2004)). Given that ARP is based on age dependent preventive maintenance periods instead of fixed periods, as in BRP, it remains more realistic and hence attracts many researchers. We refer the reader to extended versions of the age replacement policies and their implantation presented in Ajodhya and Damodar (2004). The related policies are no realistic in the context of manufacturing systems given that frequent machine breakdowns inevitably create bottlenecks for the process. Hence, preventive maintenance (to reduce likelihood of machine breakdowns) combined to the control of finished goods inventory is a potential way of reducing the overall incurred cost.

The aforementioned models are classified herein as static models given that the obtained policies are based on the mean values of the involved stochastic processes. In addition, the dynamics of the finished goods inventory is not considered in those models for a large class of manufacturing systems. Conversely, manufacturing systems with unreliable machines have been modelled using the so called stochastic optimal control theory in which failures and repairs processes were supposed to be described by homogeneous Markovian processes. The related optimal control model fails in the category of problems presented in the pioneering work of Rishel (1975). Investigation in the same direction provided the analytical solution of the one-machine, one-product manufacturing system obtained by Akella and Kumar (1986). Preventive maintenance planning problems are combined to the production control to increase the availability of the production system and hence to reduce the overall incurred cost (see Boukas and Haurie (1990)).

A preventive maintenance model for a production inventory system is developed in Das and Sarkar (1999) using information on the systems conditions (such as finished product demand, inventory position, costs of repair and maintenance, etc.) and a continuous probability distribution characterizing the machine failure process. An analytical model of BRP and safety stock strategy is formulated by Ki-Ling and Warren (1997), using also restrictive assumptions such as: the time to accomplish build-up and depletion of safety stock is small relative to the mean time to failures (MTTF). The model presented in Salameh and Ghattas (2001) combines ARP and safety stock to show that one need to built an inventory just before the preventive maintenance. It is assumed in Salameh and Ghattas (2001) that extra capacity is maintained to buffer against uncertainties of the production processes and that there is no possible breakdown of the machine before the preventive maintenance date. Without the assumption made by Salameh and Ghattas (2001) on the machine dynamics, the stochastic optimal control theory is used in Boukas et al. (1995), Gharbi and Kenne (2000, 2005), Kenne and Gharbi (1999) and in Kenne and Boukas (2003) to define an machine age dependent production and preventive maintenance policies. Such policies are based on non homogeneous Markov models, and hence are restricted to exponential distributions describing operational and down times of the involved machines.

The purpose of this paper is to investigate the joint implementation of preventive maintenance and safety stocks in a more realistic manufacturing environment using a stochastic model not restricted to markovian processes as mentioned previously. The main results presented in this paper extends the works of Ki-Ling and Warren (1997), Salameh and Ghattas (2001), Rezg et al. (2005) and references therein. Hence, we investigate herein joint implementation of preventive maintenance and safety stocks in unreliable production environment with back order (i.e., unmet demands during the repair period are not lost) and with possible failure at any age of the machine (for example during built up of the safety stock). Available models, based on reliability theory, are not able to remove the aforementioned restrictions due to the complexity of the mathematical model. Removing theses restrictions, as in this paper, involves additions concepts needed for the determination of the incurred cost expression used, here as criteria index.

The reminder of this article is organized as follows. In section 2, we define the assumptions used in the model. The probability model and the control policy are outlined in section 3. Using the properties of the probability model, we develop expressions for the measures of the system performance and present the optimality conditions in section 4. A numerical example is presented in section 5. Sensitivity and results analysis are provided in section 6 and concluding remarks are given in section 7.

2. Notations and model assumptions

Through the paper, the following notation will be used:

C^{+}	inventory holding cost per unit tine
C^{-}	penalty cost for each unit of unmet demand
C_1	cost of corrective maintenance
C_2	cost of preventive maintenance
$u(\cdot)$	production rate of the system
u_{mx}	maximal production rate
d	demand rate
T_b	random variable describing the time to machine breakdown
$f(t)$	probability density function of T_b
$F(t)$	cumulative distribution function of T_b
$r(t)$	failure rate of the machine

T_{pm}	random variable describing time to perform preventive maintenance
$q(t)$	probability density function of T_{pm}
$Q(t)$	cumulative distribution function of T_{pm}
T_{cm}	random variable describing time to perform corrective maintenance
$g(t)$	probability density function of T_{cm}
$G(t)$	cumulative distribution function of T_{cm}
S	stock threshold level or safety stock level
T	scheduled time to preventive maintenance

We consider the production of a single item on a production process with a capacity u_{mx} so as to satisfy a constant demand rate d items units per unit of time with $d < u_{\text{mx}}$. Typical examples of such a production process include stamping and press punching in the automotive industry and die casting (see Ki-Ling and Warren (1997)). The capacity u_{mx} represents the maximum production output rate, such as one where all three shifts of operation are utilized. With such a system, we first identify the stochastic process that account for all random events, namely production, failure, repair and preventive maintenance. We obtain a probability structure on those stochastic processes which are then exploited to obtain system performance measure. Failure and repair events are due to machine breakdown, which occurs in a random manner and constitute a major source of uncertainties in the production process.

Whenever a breakdown occurs, corrective maintenance is performed, during a random amount of time, to restore the machine to its initial condition (i.e., the machine is assumed to be new and its age is set to zero). During the repair periods, one of the following two situations occurs:

- demands for items are met only by safety stock;
- all unmet demands are backlogged.

Hence, the considered machine is capable of catching up with the unmet demand without interrupting the normal production process as soon as production resumes. For this situation, there is a possibility of having another breakdown during catch-up period. In order to compute the incurred cost using mean values of the involved stochastic processes, most previous models, in relevant literature, assumed that such a possibility is negligible. The likelihood of machine breakdown is reduced when preventive maintenance is scheduled and combined to production planning. Each time, immediately after the maintenance operation is performed, the machine is restored at its initial working condition.

In addition, we based the model under consideration on the following assumptions:

(A-1) The production unit is subjected to stochastic breakdowns and repairs

(A-2) If a machine failure occurs during a production phase, corrective repair is started immediately and after repair, the machine is restored back to the same initial working condition. In addition, a preventive maintenance action (as a corrective one) renews the production system (i.e., the age of the machine is set to zero).

(A-3) The mean value of the time requirement for a preventive maintenance operation is short when compared with the mean time to machine breakdown.

(A-4) The demand rate of the product is a known constant whereas the production rate (which is greater than the demand rate) depends on the decision variables S and T.

(A-5) Shortages may occur due to longer repair time. In that case, all the unsatisfied demands are backlogged

(A-6) A sufficient capacity is present to allow accumulation of safety stock in the beginning of each machine life cycle; but the time to accomplish build-up and depletion of safety stocks is not necessarily small relative to mean time to failure (MTTF) as in available models (i.e., a breakdown could arises during this time).

(A-7) There is no restriction on any of the operational, repair and preventive maintenance time distribution.

(A-8) Breakdowns of the machine don’t affect the quality of products.

3 Analytical model and control policy

Let the system state be denoted by (α, x) where α indicates the system status, and x indicates the surplus level. Note that the surplus is positive when it represents inventory and negative when it represents a shortage situation. Let (Ω, F) be a measurable space and

$\{F_t : t \geq 0\}$ an increasing class of sigma-fields representing the history of the (α, x) process. A sample value ω corresponds to an x- trajectory which is continuous, and a sequence of α-values without accumulation points. Given also the discontinuity on the machine age trajectory (set to zero after each corrective or preventive maintenance operation), the control problem in the case of joint determination of safety stock levels and machine age preventive maintenance, as in this paper, is of the type of piecewise deterministic random process. Hence, the production system under study can be considered as a deterministic system as long as no machine breakdowns or stoppage occur.

The set Γ of admissible control policies is a family of F_t-adapted processes with values in $\Pi(\alpha) = \{u(\cdot) : 0 \leq u(\cdot) \leq u_{\text{mx}}\}$. In addition, a sufficient capacity is present to allow accumulation of safety stock in the beginning of each machine life cycle. The time to accomplish build-up and depletion of safety stocks is not necessarily small relative to MTTF. Hence, there is a possibility of having another breakdown during catch-up period. The preventive maintenance policy is not implicitly bounded (i.e., not included in $\Pi(\alpha)$) given hat there is no fixed maximal preventive maintenance rate in the proposed model. The maintenance epoch is one of the control variable that we are looking for.

Let $u(\cdot) \in \Re^+$ denotes the production rate that may vary with time and with the state/capacity of the machine. Therefore, $u(\cdot) \geq 0$ and is subject to the random process $\{\alpha(t), t \geq 0\}$. With the total surplus $x(\cdot) \in \Re$ and the demand rate $d \in]0, \infty)$, the continuous part of the system dynamics is described by the following differential equation:

$$\frac{dx(t)}{dt} = u(t) - d \qquad x(0) = x \tag{1}$$

where $x \in \Re$ is the initial surplus level. Recall that all unmet demands are backlogged and a penalty cost is incurred per item on a per unit basis.

The discrete part of the system dynamics is described by the system status $\alpha(t) \in \{1, 2, 3\}$ with $\alpha(t) = 1$ if the machine is operational, $\alpha(t) = 2$ if the machine is under repair and $\alpha(t) = 3$ if the machine is under preventive maintenance. The machine state moves from the different modes of the process $\alpha(t)$ according to random variables T_b, T_{cm} and T_{pm} defined as time to machine breakdown, corrective maintenance time and preventive maintenance time. At mode 1 and for a given age, the production rate is given by an extended version of the so called hedging point policy (HPP) defined by a threshold level S. Such a policy is given by:

$$u(x) = \begin{cases} u_{\text{mx}} & \text{if } x < S \\ d & \text{if } x = S \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

In such a policy, an optimal inventory level S is maintained during time of excess capacity availability to hedge against future capacity shortage brought by machine failures. If the current inventory level exceeds the optimal inventory level S, one should not produce at all; if less, one should produce at the maximum rate u_{mx}; while if exactly equal, one should exactly produce enough to meet demand d.

The ARP is combined to the hedging point policy, defined by equation (2), and the result leads to the proposed control policy. Recall that the ARP consists of a preventive maintenance which is performed at a scheduled time that depends on the age of the machine. The proposed policy hence depends on two parameters, namely S and T, and is completely defined for given values of those parameters. While producing, the machine and surplus dynamic both involve different scenarios used in the next section to develop optimality conditions from which values of the proposed control policy parameters are determined.

4. Optimality conditions

The expected cost per unit of time, used herein as optimisation performance criteria, includes the surplus costs, the preventive and corrective maintenance costs. Note that surplus costs consist of inventory cost for positive surplus and backlog cost for negative surplus.

4.1. Failure and surplus costs

The inventory and backlog costs are determined through the investigation of two different scenarios based on the fact that the inventory level reaches the optimal inventory level or not.

<u>*Scenario no. 1*</u>*:* There is a breakdown during the building of a safety stock S at the rate $u_{mx} - d$ and the involved repair process ended with inventory or at a positive surplus level. The finished goods inventory in such a situation is illustrated in figure 1. The mean time to repair the machine without backlog is given by:

$$a_r - a_f = tmw_1 = \int_0^{t_1} t \cdot g(t) \cdot dt \qquad (3)$$

where $t_1 = \dfrac{a_f \cdot (u_{mx} - d)}{d}$, a_f is the failure age and a_r is the ending age related to the repair process.

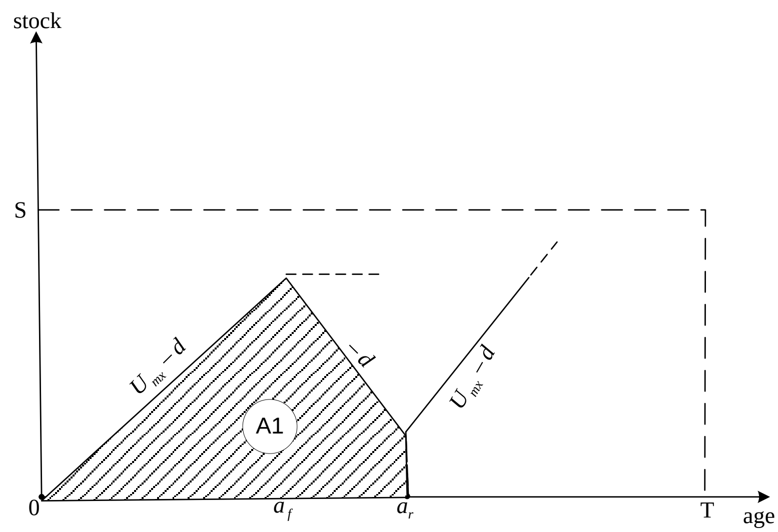


Figure 1: Surplus sample path for a failure during the phase of safety building with inventory at the end of the repair process.

Using the dashed areas illustrated on figure 1, the inventory before and after the failure is represented by $Surf_{INV}^{1}$:

$$Surf_{INV}^{1} = A1 = \int_{T_1}^{T_2} \left(a_f^2 \cdot \frac{(u_{mx} - d)}{2} + \frac{tmw_1}{2} \cdot \left(2 \cdot a_f \cdot (u_{mx} - d) - tmw_1 \cdot d\right) \right) \cdot f(a_f) da_f \qquad (4)$$

where $T_1 = 0$ and $T_2 = \dfrac{S}{U_{mx} - d}$. The repair process ended with inventory if the involved repair time is less than $a_f \cdot (u_{\text{mx}} - d)/d$.

<u>Scenario no. 2</u>*:* There is a breakdown during the building of a safety stock S at the rate $u_{\text{mx}} - d$ and the involved repair process ended with backlog or at a negative surplus level. The surplus in such a situation is illustrated in figure 2.

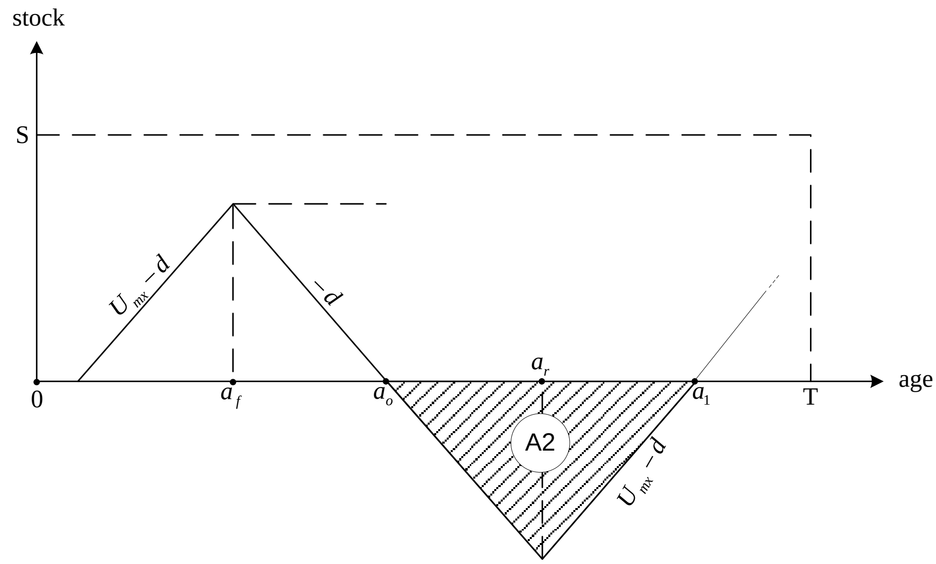


Figure 2: Surplus sample path for a failure during the phase of safety building with backlog at the end of the repair process.

In this scenario, the failure occurs at the age a_f . The repair time exceeds

$a_f \cdot (u_{\max} - d)/d$ and hence involves integration from a_f *to* ∞ as in equation (5). The backlog is represented by the surface $Surf_{Bkg}^{2}$:

$$Surf_{Bkg}^{2} = A2 = \int_{T_1}^{T_2} \left\{ \left(\int_{a_f \cdot \frac{(u_{mx}-d)}{d}}^{\infty} \left(t - a_f \cdot \frac{(u_{mx}-d)}{d} \right) \cdot g(t)\,dt \right)^2 \cdot \frac{d}{2} \cdot \left(1 + \frac{d}{(u_{mx}-d)} \right) \right\} \cdot f(a_f)\,da_f \quad (5)$$

The cost for a failure during the phase of safety building is given by equation (6) which is obtained by grouping equations (4) and (5) from scenarios 1 and 2 (i.e.,

$Cost_{12} = C^{+} \cdot Surf_{INV}^{1} + C^{-} \cdot Surf_{Bkg}^{2}$). Note that obtained expression of the cost is multiplied by $f(a_f)da_f$ which is the probability to have a failure at age a_f.

$$Cost_{12} = \int_{0}^{\frac{s}{Umx-d}} \left\{ \begin{array}{l} C^{+} \cdot \left[\begin{array}{l} a_f^2 \cdot \dfrac{(u_{mx}-d)}{2} + \\ \dfrac{tmw_1}{2} \cdot \left(2 \cdot a_f \cdot \dfrac{(u_{mx}-d)}{d} - tmw_1 \cdot d \right) \end{array} \right] + \\ C^{-} \cdot \left(\int_{a_f \cdot \frac{(u_{mx}-d)}{d}}^{\infty} \left(t - a_f \cdot \dfrac{(u_{mx}-d)}{d} \right) \cdot g(t)\, dt \right)^2 \cdot \dfrac{d}{2} \cdot \left(1 + \dfrac{d}{(u_{mx}-d)} \right) \end{array} \right\} \cdot f(a_f)\, da_f \qquad (6)$$

Scenario no. 3: There is a failure at the saturation phase, in which the stock level is kept at level S and the production rate is reduced to d, and the involved repair process ended with inventory or at a positive surplus level. The finished goods inventory in such a situation is illustrated in figure 3.

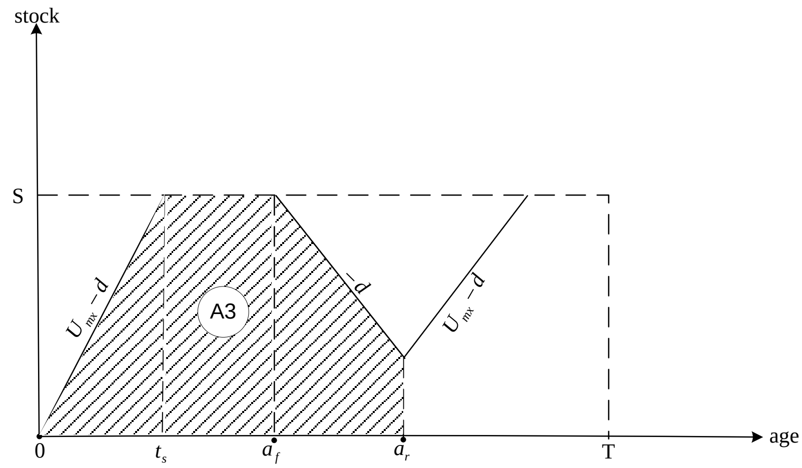


Figure 3: Surplus sample path for a failure during the saturation phase with inventory at the end of the repair process

The failure occurs at a machine age located between $t_s = S/(u_{mx}-d)$ and the scheduled preventive maintenance time T. Note that t_s is the age at which the safety stock levels is reached. The mean operational time of the machine is described by $m(T)$ given by:

$$m(T) = \int_0^T t \cdot f(t) \cdot dt + R(T) \cdot T \qquad (7)$$

The mean time to repair of the machine without backlog is described by tmw_2 and given by:

$$tmw_2 = a_r - a_f = \int_0^{\frac{s}{d}} t \cdot g(t) \cdot dt \tag{8}$$

The inventory surface related scenario 3 (i.e., dash area $A3$ illustrated in figure 3), is given by:

$$Surf_{INV}^{3} = A3 = S \cdot m(T) + \left(2 \cdot S - tmw_2 \cdot d\right) \cdot \frac{tmw_2}{2} \tag{9}$$

<u>*Scenario no. 4*</u>*:* There is failure at the saturation phase and the involved repair process ended with backlog or at a negative surplus level. The surplus in such a situation is illustrated in figure 4.

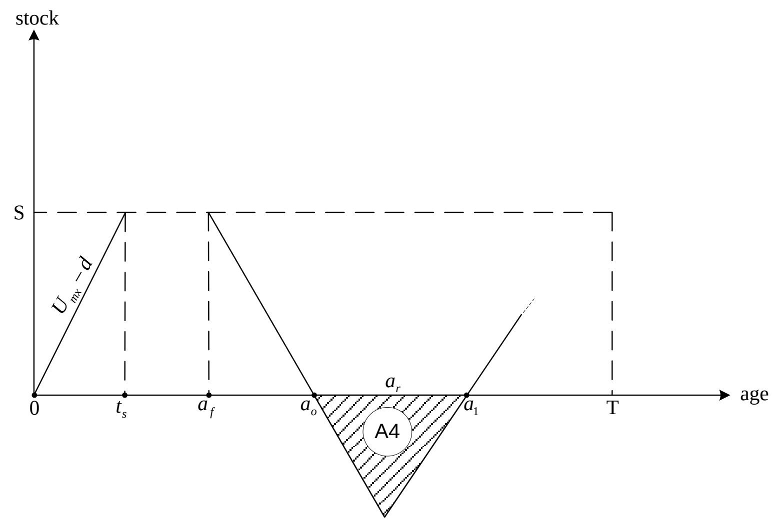


Figure 4: Surplus sample path for a failure during the saturation phase with backlog at the end of the repair process.

The mean time to repair of the machine with backlog (i.e., repair time greater than S/d) is described by tmw_3 and given by:

$$tmw_3 = a_r - a_f = \int_{\frac{s}{d}}^{\infty} \left(t - \frac{S}{d}\right) \cdot g(t) \cdot dt \tag{10}$$

The backlog surface related to scenario 4 (i.e., dash area $A4$ illustrated in figure 4), is given by:

$$Surf_{Bkg}^{4} = A4 = (tmw_3)^2 \cdot \frac{d}{2} \cdot \left(1 + \frac{d}{(u_{mx} - d)}\right) \tag{11}$$

Given that the failure occurs after $t_s = S/(u_{mx} - d)$ and before T, equation (11) is rewritten using $F(x)$ which is the cumulative distribution function of the time to machine breakdown evaluated at a_f. One obtain the following expression:

$$Surf_{Bkg}^{4} = \left[F(T) - F\left(\frac{S}{(u_{mx} - d)} \right) \right] \cdot \left(\int_{\frac{s}{d}}^{\infty} \left(t - \frac{S}{d} \right) \cdot g(t) \cdot dt \right)^2 \cdot \frac{d}{2} \cdot \left(1 + \frac{d}{(u_{mx} - d)} \right) \quad (12)$$

The cost for a failure during the saturation phase is given by equation (13) which is obtained by grouping equations (9) and (11) from scenarios 3 and 4 (i.e., $C\,\mathrm{os}\,t_{34} = C^{+} \cdot Surf_{INV}^{3} + C^{-} \cdot Surf_{Bkg}^{4}$).

$$\begin{aligned} Cost_{34} = {} & C^{+} \cdot \left[S \cdot m(T) + (2 \cdot S - tmw_2 \cdot d) \cdot \frac{tmw_2}{2} \right] + \\ & C^{-} \cdot \left[F(T) - F\left(\frac{S}{(u_{mx} - d)} \right) \right] \cdot \left(\int_{\frac{s}{d}}^{\infty} \left(t - \frac{S}{d} \right) \cdot g(t) \cdot dt \right)^2 \cdot \frac{d}{2} \cdot \left(1 + \frac{d}{(u_{mx} - d)} \right) \end{aligned} \quad (13)$$

4.2. Preventive maintenance and surplus costs

The proposed model includes preventive actions, described by the distribution function $f_p(\cdot)$. Two different scenarios are considered using the surplus sign (positive for inventory and negative for backlog) at the end of the preventive maintenance action.

<u>*Scenario no. 5*</u>**:** The preventive maintenance action starts at T and ends with inventory or at a positive surplus level as illustrated in figure 5.

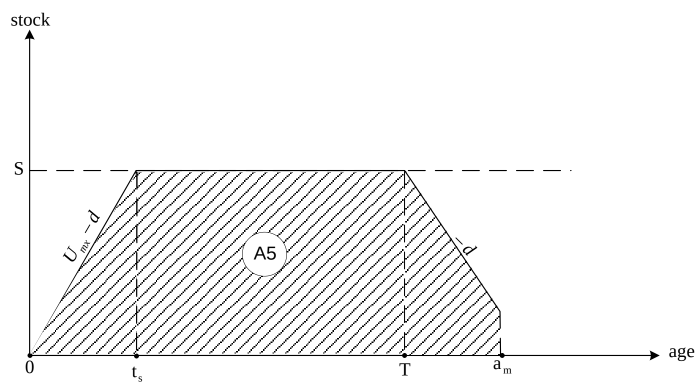


Figure 5: Surplus sample path for a preventive maintenance at T with inventory at the end of the preventive maintenance process

The mean time to preventive maintenance without backlog (i.e., maintenance time less than S/d) is described by t_{pm}^{+} and given by:

$$a_m - T = t_{pm}^{+} = \int_0^{\frac{s}{d}} t \cdot q(t) \cdot dt \qquad (14)$$

where a_m is the ended age of the preventive maintenance process. The inventory surface related scenario 5 (i.e., dash area $A5$ illustrated in figure 5), is given by:

$$Surf_{INV}^{5} = A5 = S(T - \frac{t_s}{2}) + \left(2 \cdot S - t_{pm} \cdot d\right) \cdot \frac{t_{pm}^{+}}{2} \qquad (15)$$

<u>*Scenario no. 6*</u>*:* The preventive maintenance action starts at T and ends with backlog or at a negative surplus level as illustrated in figure 6.

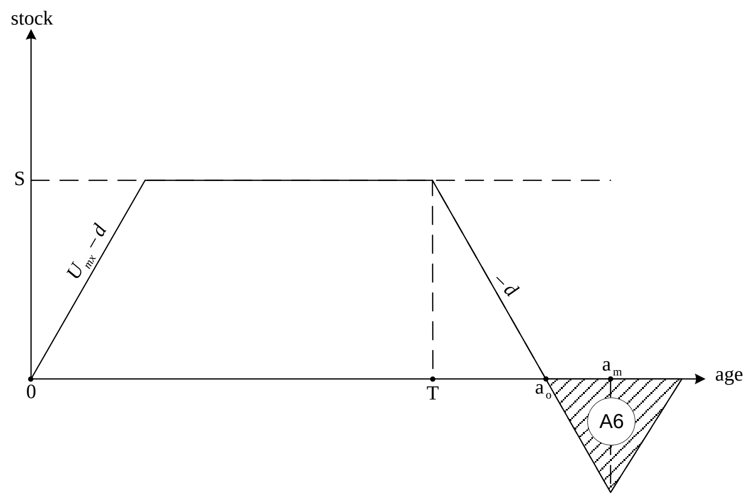


Figure 6: Surplus sample path for a preventive maintenance at T with backlog at the end of the preventive maintenance process

The mean time to preventive maintenance with backlog (i.e., maintenance time greater than S/d) is described by t_{pm}^{-} and given by:

$$t_{pm}^{-} = a_m - T = \int_{S/d}^{\infty} (t - \frac{S}{d}) \cdot q(t) \cdot dt \qquad (16)$$

The backlog surface related scenario 6 (i.e., dash area $A6$ illustrated in figure 6), is given by:

$$Surf_{Bkg}^{6} = A6 = \left(t_{pm}^{-}\right)^2 \cdot \frac{d}{2} \cdot \left(1 + \frac{d}{\left(u_{mx} - d\right)}\right) \qquad (17)$$

The cost due to preventive maintenance at T is obtained by grouping equations (15) and (17) from scenarios 5 and 6 (i.e., $C\mathrm{os}t_{56} = C^{+} \cdot Surf_{INV}^{5} + C^{-} \cdot Surf_{Bkg}^{6}$). The obtained sum is multiplied by $R(T)$ which is the probability to go up to preventive maintenance.

$$Cost_{56} = R(T) \cdot \left\{ \begin{array}{l} C^{+} \cdot \left[S(T - \frac{t_s}{2}) + \left(2 \cdot S - t_{pm} \cdot d\right) \cdot \frac{t_{pm}}{2} \right] + \\ C^{-} \cdot \left(\int_{\frac{s}{d}}^{\infty} \left(t - \frac{S}{d} \right) \cdot q(t) \cdot dt \right)^{2} \cdot \frac{d}{2} \cdot \left(1 + \frac{d}{\left(u_{mx} - d\right)} \right) \end{array} \right\} \tag{18}$$

4.3. Overall cost and optimality conditions

The production cost includes corrective and preventive maintenance costs obtained by multiplying the involved penalty by the occurrence probability. The cost generated by the preventive maintenance action is obtained by multiplying the preventive maintenance cost (i.e., C_2) by the probability of its occurrence (i.e., $R(T)$ which is the reliability function of the machine evaluated at T). The cost generated by the corrective maintenance action is also obtained by multiplying the corrective maintenance cost (i.e., C_1) by the probability of its occurrence (i.e., $F(T)$ which is the failure cumulative function of the machine evaluated at T). The maintenance cost (corrective and preventive) is then given by:

$$CM = R(T) \cdot C_2 + F(T) \cdot C_1 \tag{19}$$

According to assumption (A-2) and equation (7), the duration of a production cycle is approximated given by:

$$Cycle\_Time = m(T) + R(T) \cdot MTTP + F(T) \cdot MTTR \tag{20}$$

where $MTTP$ and $MTTR$ are mean time to preventive maintenance and mean time to repair. The overall expected cost is obtained by summing scenarios costs (equations (6), (13) and (18)) and maintenance costs given by equation (19). By dividing the obtained sum by the cycle time given by equation (20), one obtains the overall expected cost per unit time $L(\cdot)$, given by the following equation:

$$L(S,T) = \frac{Cost_{12} + Cost_{34} + Cost_{56} + CM}{m(T) + R(T) \cdot MTTP + F(T) \cdot MTTR} \tag{21}$$

The optimality conditions and hence the age dependent optimal value of the safety stock (threshold level) and preventive maintenance epoch are given by the following equations:

$$\left.\frac{\partial L(S,T)}{\partial S}\right|_T = 0 \tag{22}$$

$$\left.\frac{\partial L(S,T)}{\partial T}\right|_S = 0 \tag{23}$$

Due to the complexity of previous expressions (see equation (21) and dependent equations), proving the convexity of $L(S,T)$ and obtaining the analytical optimality conditions from equations (22) and (23) become more complex. Hence, instead of solving (22) and (23) to obtain optimal values of the involved parameters, a numerical procedure, using a simple enumeration is presented in the next section. The best feasible solution (S^*,T^*) is given further from the application of the proposed numerical procedure.

5. Numerical procedure and example

The following iterative numerical procedure has been developed to find the optimal control policy, characterized by parameters S^* and T^*, and the optimal overall incurred cost.

Input: $u_{mx}, d, C^+, C^-, C_1, C_2, f(t), f_r(t), f_p(t), S_{\min}, S_{\max}, T_{\min}, T_{\max}, MTTR, MTTP, T_{inc}, S_{inc}$

Step 2: Set $S := S_{\min}$ and $T := T_{\min}$

Step 3: Compute tmw_i, $(i = 1,2,3)$, and tpm^+, tpm^- using equations (3), (8), (10), (14) and (16)

Step 4: Compute $Cost_{12}(S)$ using equation (6)

Step 5: Compute $m(T)$, $Cost_{34}(S,T)$, $Cost_{56}(S,T)$ and $CM(T)$ using equations (7), (13), (18) and (19)

Step 6: Compute the total cost $L(S,T)$ using equation (21)

Step 7: If $T < T_{\max}$ then set $T := T + T_{inc}$ and go to Step 5; else go to Step 8

Step 8: If $S < S_{\max}$ then set $S := S + S_{inc}$ and go to Step 3; else find the solution that minimise $L(S,T)$

Output: S^* *and* T^* = optimal threshold level and time for preventive maintenance

$L(S^*, T^*)$ = optimal cost

Stop

The previous numerical scheme proceeds as follows:

a) read input data
b) consider computational grid on T and S for given lower and upper bounds $(T^{\min}, T^{\max})$ and $(S^{\min}, S^{\max})$ respectively (see step 2).
c) Compute the overall cost (see steps 3 to 6)
d) for each feasible schedule preventive maintenance time T (i.e., $T^{\min} \le T \le T^{\max}$), consider a discrete time interval T_{inc} and solve the optimality condition at time t to obtain the optimal cost and the associated threshold level (see step 7)
e) for each feasible schedule threshold level S (i.e., $S^{\min} \le S \le S^{\max}$), consider a discrete stock interval S_{inc} and solve the optimality condition at time t to obtain the optimal cost and the associated preventive maintenance time (see step 7)
f) return the lowest cost and the associated S and T called hereinafter optimal threshold and preventive maintenance time (i.e., $L(S^*, T^*)$, S^* *and* T^*).

We consider a fairly general example problem as a vehicle for providing further details on the solution of the optimisation control problem under study. Also presented are numerical results that provide further insight to the problem. For an illustrative purpose, assume $u_{mx} = 1$ item per unit of time and the production process is run to satisfy a constant demand rate $d = 0.65$ item per unit of time. In addition, the following parameters are adopted for the basic case (different others cases are considered later during a sensitivity analysis): $C_1 = \$5000$, $C_2 = 3000$, $C^+ = 5$, $C^- = 50$. The time to breakdown T_b is Weibull with $\beta = 2$ and η=100 (i.e., $\mu = 88.6$). The time to repair and the time to preventive maintenance are lognormal with μ_{cm}=10, $\sigma_{cm} = 1$ and μ_{pm}=5, $\sigma_{pm} = 0.5$ respectively.

Using the proposed iterative numerical procedure and the previous data, we obtain an overall cost function represented by its contour plot in figure 7. The optimal cost for this example is $L^*(\cdot) = \$87$ and the corresponding control parameters are: $S^* = 2.7$ and $T^* = 67$.

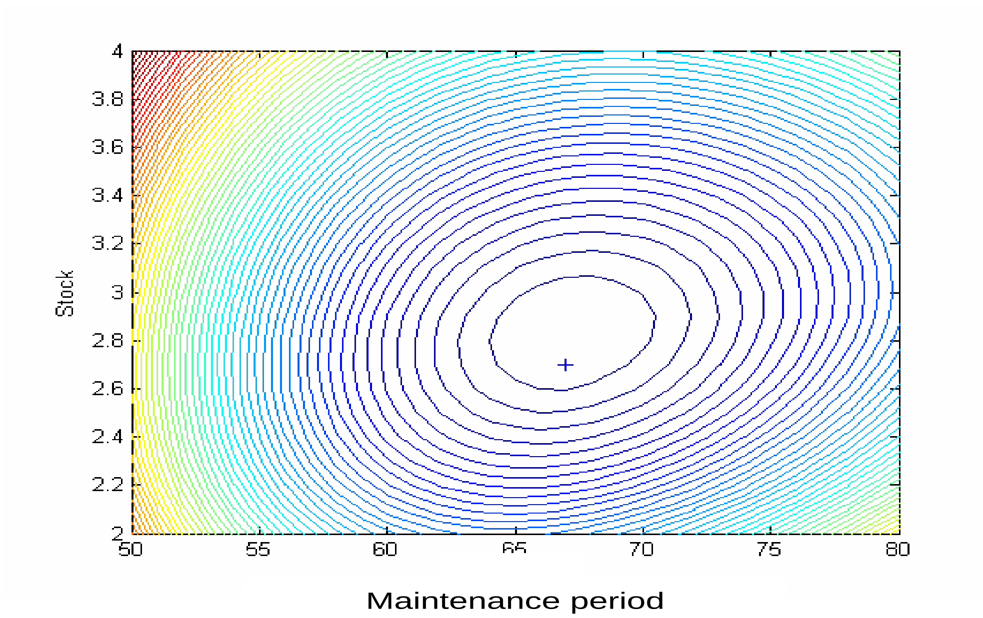


Figure 7: Contour plot of the overall cost function and Optimal control policy parameters for the illustrative example

The next section presents the robustness of the developed model through a sensitivity analysis and illustrates the usefulness of the approach proposed.

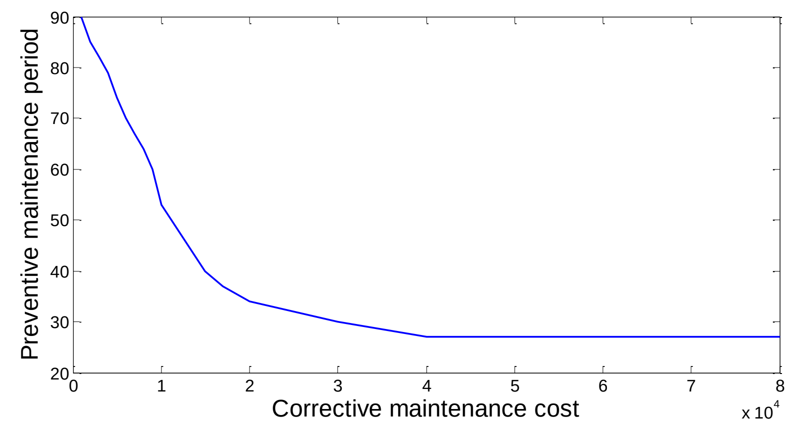
6. Sensitivity analysis

Four classes of studies are considered through the variations of corrective maintenance, preventive maintenance, inventory and backlog, costs. For those classes, we illustrate the sensitivity analysis through figures 8 to 11.

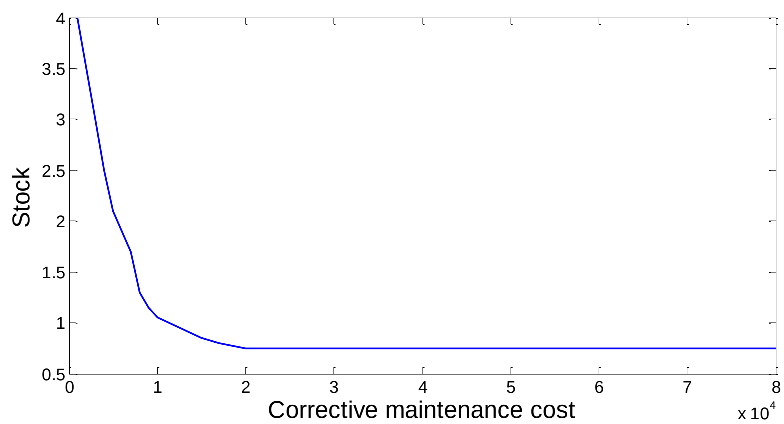
It is interesting to note the following from figure 8, obtained with the variation of the corrective maintenance cost C_1 from \$3000 to \$80000 with $C_2 = \$3000$, $C^+ = \$5$ and $C^- = \$50$:

- The scheduled production time for preventive maintenance T^* decreases with the increasing of the corrective maintenance cost due to the fact that the safety stock decreases; and hence increase the possibility to have a backlog situation (see figure 8(a)).
- The optimal threshold level decreases with the increasing of the corrective maintenance cost and converges to an asymptotic value for large values of such a cost as the scheduled preventive maintenance time (see figures 8(a) and 8(b)).

It is clear from figure 8 that the corrective maintenance cost has a significant influence on the optimal threshold level and the scheduled preventive maintenance time and hence to the overall incurred cost.



(a)



(b)

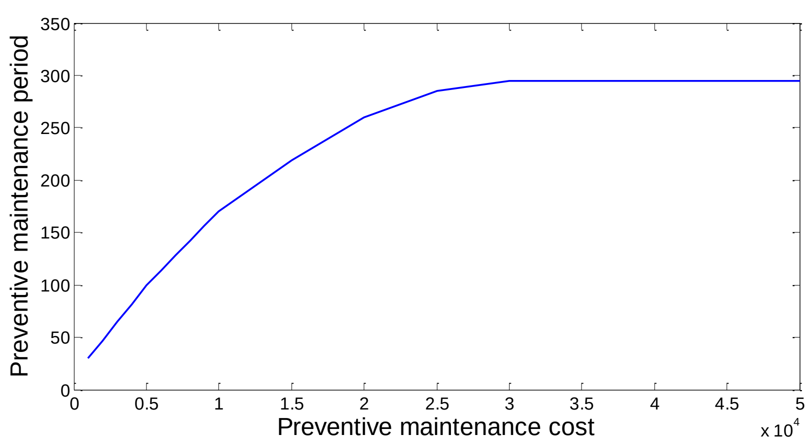
Figure 8: Optimal production and preventive maintenance policies for different corrective maintenance costs.

The asymptotic behaviour observed both in figures 8(a) and 8(b) states that for large value of the corrective maintenance cost, the preventive maintenance period attempts a minimal value that results to a small value of the stock level. This is due to the fact that a minimal value of preventive maintenance period corresponds to more frequent preventive maintenance actions that avoid failures for which excessive costs are considered.

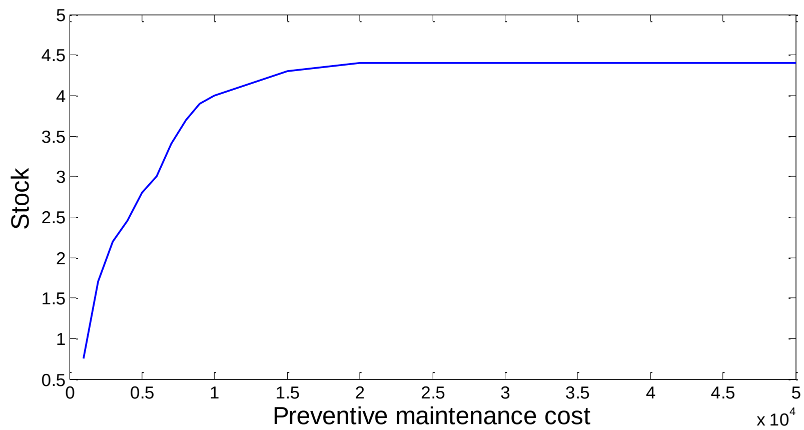
From figure 9, obtained with the variation of the preventive maintenance cost C_2 from $\$100$ to $\$50000$ with $C_1 = \$5000$, $C^+ = \$5$ and $C^- = \$50$, we note the following:

- The scheduled production time for preventive maintenance T^* increases with the increasing of the preventive maintenance cost given that one need to reduce the frequency of preventive maintenance due to their excessive cost (i.e., for large values of C_2). Such a structure is illustrated in figure 9(a).
- The optimal threshold level increases with the increasing of the preventive maintenance cost and converges to an asymptotic value for large values of such a cost as the scheduled preventive maintenance time (see figure 9(b)).

The lower frequency of preventive maintenance at large values of C_2 increases the breakdown frequency of the machine. To hedge against possible disruption of the inventory due to failures, large values of threshold levels are recommended as shown by the obtained results (see figure 9(b)).



(a)



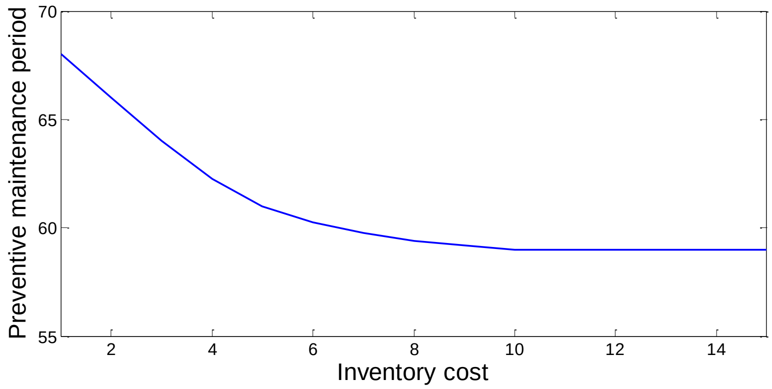
(b)

Figure 9: Optimal production and preventive maintenance policies for different preventive maintenance costs

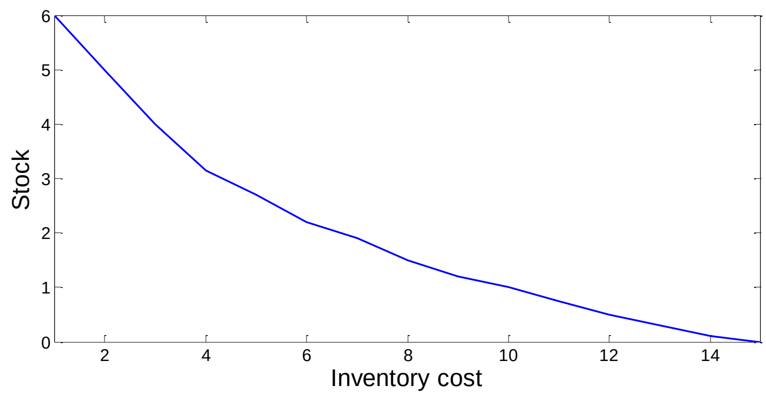
It is interesting to note an asymptotic behaviour of both preventive maintenance period and stock level from figure 9. Such a behaviour, is due to the interaction between the preventive maintenance period and the stock level which obviously depends on the MTTR and MTTF of the considered distributions.

From figure 10, obtained with the variation of the inventory cost C^+ from \$1 to \$15 with $C_1 = \$5000$, $C_2 = \$3000$ and $C^- = \$50$, we note the following:

- The scheduled production time for preventive maintenance T^* decreases with the increasing of the inventory cost due to the fact that more frequent preventive maintenance is needed to avoid excessive inventory levels (i.e., not recommended for large values of C^+). Note that T^* converges also to an asymptotic value for large values of the inventory cost (see figure 10(a))
- The optimal threshold level decreases with the increasing of the inventory cost and converges to zero for large values of such a cost as shown in figure 10(b).



(a)



(b)

Figure 10: Optimal production and preventive maintenance policies for different inventory costs

The higher frequency of preventive maintenance at large values of C^{+} decreases the breakdown frequency of the machine. Hence there is no need to keep a significant safety stock level for large values of C^{+} (i.e., $S^{*} \rightarrow 0$ as $C^{+} \rightarrow \infty$).

From figure 11, obtained with the variation of the inventory cost C^{-} from \$20 to \$400 with $C_1 = \$5000$, $C_2 = \$3000$ and $C^{+} = \$5$, we note the following:

- The scheduled production time for preventive maintenance T^{*} decreases with the increasing of the backlog cost to increase the availability of the production system through frequent preventive maintenance at large values of C^{-} (see figure 11(a)).
- The optimal threshold level increases with the increasing of the backlog cost and converges to an asymptotic value for large values of such a cost as the scheduled preventive maintenance time (see figure 11(b)).

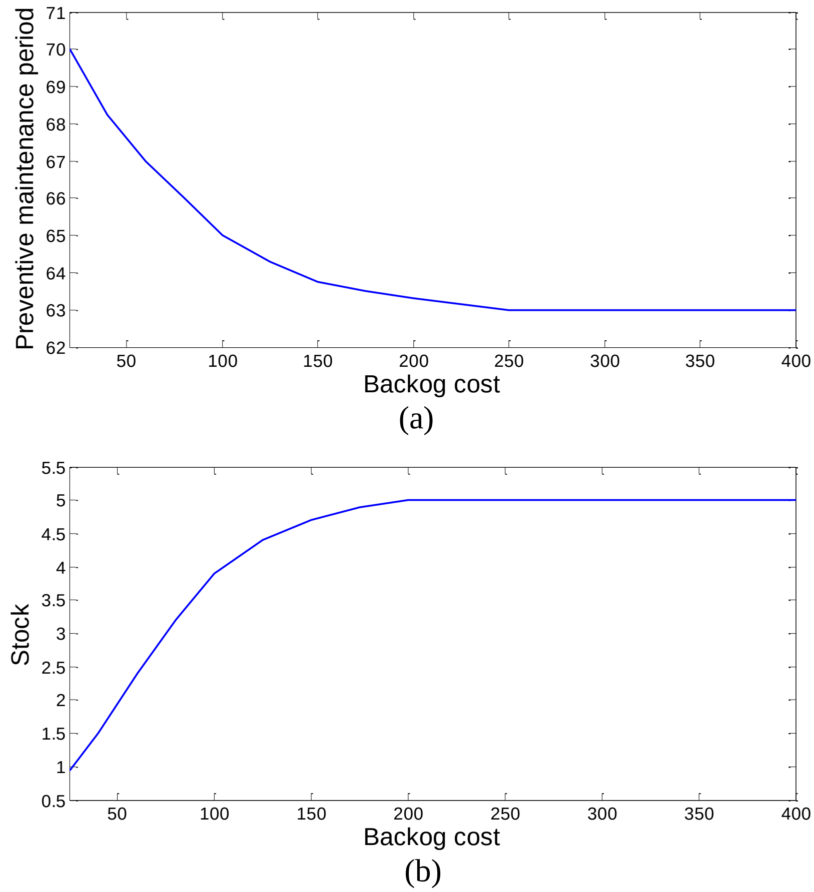


Figure 11: Optimal production and preventive maintenance policies for different backlog costs

As previously, the higher frequency of preventive maintenance at large values of C^{-} decreases the breakdown frequency of the machine and the relatively small value of the inventory cost ensure to the system a comfortable inventory level to hedge against potential failures that could generate backlog and excessive cost.

The results presented in this paper indicate that, as expected, the optimal production policy for the considered manufacturing system is characterized by two parameters namely optimal threshold level S^{*} and scheduled preventive maintenance period T^{*}. The control policy (2) is completely defined by values S^{*} and T^{*}

The trends of the curves shown in figures 8 to 11 confirm the robustness of the proposed approach through a sensitivity analysis. This is performed by threshold levels and scheduled preventive maintenance periods versus an overall incurred cost including inventory, backlog, corrective and preventive costs. The asymptotic behaviour, well illustrated in figures 8 to 11, shows clearly that obtained results make sense and that the proposed approach is robust.

For the system considered previously and related to a one-machine, one-product manufacturing system, the production and preventive maintenance policies are completely known for given parameters S and T. For a more complex manufacturing system consisting of m machines producing n different part types, the production and preventive maintenance policies depend on the parameters $T_1,\cdots,T_m$ and $S_1,\cdots,S_n$. As a result, $m+n$ parameters or factors could be used to define the control policy in the context of a multiple parts, multiple products manufacturing system. The experimental design approach, combined to simulation and analytical models could be used to determine the effects of considered factors on the incurred costs and to determine their optimal values. Details on experimental design and simulation modelling could be find in Gharbi and Kenne (2000).

7. Conclusion

A production inventory and preventive maintenance system with general characteristics and realistic assumptions has been considered here. The primary objective of the study was to determine when to perform the preventive maintenance, if any, on the system and the level of the safety stock so as to improve the system performance (i.e., the overall incurred cost). The mathematical model of the system provided an useful tool for deriving the expressions for the system performance measure. It was demonstrated, through a numerical example problem, how the cost based measure can be used as a basis of determining optimal level of the safety stock and the scheduled preventive maintenance period. As a result we were able to use a numerical search method to locate the optimum point (i.e., optimum parameters for production and preventive maintenance). The randomness involved in various operational aspects of the system makes it fairly difficult to analyse.

Furthermore, our assumptions of general probability distributions for all of the associated random variables (except the time between demand arrived, assumed constant) make the analysis of the system more involved. In the absence of closed form for the incurred cost, we used numerical methods to evaluate and hence to determine optimal values of the control parameters.

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