Proceedings of the Canadian Society for Mechanical Engineering International Congress
32nd Annual Conference of the Computational Fluid Dynamics Society of Canada
Canadian Society of Rheology Symposium
CSME-CFDSC-CSR 2025
May 25–28, 2025, Montréal, Québec, Canada

From Eshelby's Solution to Topological Derivatives in General Elasticity Theory

Yi Cui*
[1] Department of Mechanical and Aerospace Engineering, Nagoya university, Nagoya, Japan
*yi.cui@mae.nagoya-u.ac.jp

ABSTRACT

The concept of topological derivatives has emerged as a powerful tool in structural optimization, allowing for the systematic identification of regions where material can be added or removed to improve performance. This paper presents a rigorous derivation of topological derivatives in general elasticity theory, with a particular emphasis on Eshelby's solution for inhomogeneities in an elastic medium. Eshelby's tensor, originally introduced to describe the elastic field perturbations due to an embedded inclusion, serves as the foundation for the asymptotic expansion used in topological sensitivity analysis. We begin by revisiting Eshelby's classical solution and its implications for elasticity problems involving inhomogeneous materials. The fundamental role of the polarization tensor in describing the perturbations caused by small inclusions is examined in detail. We then extend this framework to analyze the topological derivative, which quantifies the sensitivity of a given objective function to infinitesimal material modifications. The derivation proceeds through an asymptotic expansion approach, wherein the leading-order term captures the first-order effect of a vanishingly small inclusion on the structural response. To validate the theoretical framework, we present numerical implementations based on the finite element method (FEM) and the numerical evaluation of polarization tensors. The numerical procedures leverage FreeFem++ and Octave to compute the topological derivatives in both isotropic and anisotropic elastic media. We compare these numerical results with analytical predictions, demonstrating a high degree of agreement and underscoring the robustness of the proposed approach.

A key contribution of this study is the explicit derivation of topological derivatives for three-dimensional anisotropic materials, where existing approaches have been primarily limited to isotropic cases. The results provide new insights into the impact of material anisotropy on the sensitivity functions and open new possibilities for optimization in advanced composite materials and metamaterials. Finally, we discuss the potential applications of topological derivatives in practical engineering design, including structural optimization, damage detection, and computational material science. The presented methodology establishes a bridge between classical Eshelby theory and modern topological optimization techniques, offering a unified perspective on material redistribution in elasticity problems. Future work will extend this approach to nonlinear elasticity and multiphysics problems, further broadening its applicability in engineering and applied sciences.

Keywords: Topological derivative, Eshelby tensor, polarization tensor, structural optimization, anisotropic elasticity, finite element method, FreeFem++.