Proceedings of the Canadian Society for Mechanical Engineering International Congress
32nd Annual Conference of the Computational Fluid Dynamics Society of Canada
Canadian Society of Rheology Symposium
CSME-CFDSC-CSR 2025
May 25–28, 2025, Montréal, Québec, Canada

DEVELOPMENT OF OPTIMAL LUMPED-PARAMETER MODELS FOR TORSIONAL VIBRATION OF SHAFTS

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Abstract— This study proposes a numerical procedure for developing optimal lumped-parameter models (LPMs) for the torsional vibration of shafts. The procedure can ensure that the natural frequencies of an *n*-degree-of-freedom (*n*-DOF) LPM closely match the first *n*-1 torsional natural frequencies (TNFs) of the given shaft. As a starting point, an analytical solution is sought using a shaft with a uniform diameter, demonstrating that a 2-DOF LPM can be constructed to match the first two TNFs exactly. To tackle shafts with variable diameters and to match more than 2 TNFs, a numerical procedure based on nonlinear optimization techniques is developed. Through several case studies, the proposed procedure is shown to produce LPMs that accurately replicate the lower-order TNFs of shafts with complex geometries.

Keywords-torsional vibration of shafts; lumped-parameter model; torsional natural frequencies; nonlinear optimization

I. INTRODUCTION

Reciprocating compressors are extensively used in the energy sector, where they play a vital role in transporting oil and gas from producers to consumers. A key subsystem within these machines is the drivetrain, which comprises the coupling, crankshaft, connecting rod, and piston. Due to the inherently cyclic loading in reciprocating compressors, the crankshaft is subjected to torsional vibration. These periodic angular twisting can lead to the initiation and growth of fatigue cracks, ultimately resulting in crankshaft failure [1-3]. To prevent such failures and ensure reliable operation, torsional vibration analysis (TVA) is essential and is explicitly mandated by the American Petroleum Institute (API) standards [4].

A TVA model for a crankshaft can be developed using several methods, such as finite element method (FEM) [5, 6], the transfer matrix method (TMM) [7, 8] or the lumped-parameter model (LPM). In LPM, the crankshaft's distributed mass and inertia are replaced by a finite number of lumped masses or rigid bodies, interconnected by massless elastic and damping elements. For TVA, LPM is often preferred because it results in a diagonal mass matrix, which simplifies both modal and transient response analyses [9–11]. The accuracy of TVA, however, depends heavily on the fidelity of the LPM used.

In practice, manufacturers typically provide the mass-elastic data required for LPMs without disclosing how these values are derived. When such data is unavailable, the crankshaft can be discretized into multiple segments. For simple geometries like shafts, torsional stiffness and mass moments of inertia can be calculated analytically. For more complex geometries, such as crank throws, commercial finite element (FE) packages are commonly used. A widely adopted approach is to model short, high-inertia components like crank pins, webs, and flywheels as lumped inertia nodes, while long, low-inertia components like the crankshaft sections are treated as massless springs [10, 11]. This approach produces an approximate LPM; however, no standardized guideline currently exists for how to properly distribute these lumped inertias.

In this study, we propose a systematic procedure for developing an optimal LPM for the torsional vibration analysis of shafts. The shafts considered have free-free boundary conditions, resulting in one rigid mode and an infinite number of torsional modes. For a given shaft, an optimal *n*-degree-of-freedom (*n*-DOF) LPM is defined as one whose natural frequencies closely match the first *n*-1 torsional natural frequencies (TNFs) of the actual shaft. We begin by exploring the feasibility of deriving an analytical solution to this problem. Subsequently, a numerical procedure is formulated using nonlinear optimization techniques. Through several case studies, we demonstrate that the proposed procedure can effectively construct an optimal LPM for complex crankshafts where analytical TNFs are not available.

II. ANALYTICAL SOLUTION

First, an attempt is made to develop an LPM for a shaft with a uniform diameter as shown in Fig. 1(a) analytically. A 2-DOF LPM (one rotational DOF and one torsional DOF) shown in Fig. 1(b) is used to approximate the shaft so that its natural frequency matches the first TNF of the shaft.

For a free-free rod, its TNFs are given by [12]

$$\omega_i = \frac{i\pi}{l}\sqrt{\frac{G}{\rho}},\ i = 1, 2, \cdots \quad (1)$$

where G and ρ are the shear modulus and density of the shaft material, respectively. On the other hand, the natural frequency of the 2-DOF LPM is given by

$$\omega_n = \sqrt{\frac{k}{I_{eq}}} \quad (2)$$

where I_{eq} is the equivalent mass moment of inertia and $k = GJ/l$ is the torsional stiffness with J as the polar moment of inertia of the cross-sectional area and. A natural choice for I_e is to divide the total mass moment I of inertia into half or $I_e = I/2$. Then

$$I_{eq} = \frac{I_e}{2} = \frac{I}{4}. \quad (3)$$

Substituting k and I_{eq} into (2) and invoking the relation of $I = \rho J l$ yield

$$\omega_n = \sqrt{\frac{GJ/l}{I/4}} = \frac{2}{\pi}\frac{\pi}{l}\sqrt{\frac{G}{\rho}} = 0.6365\omega_1 \quad (4)$$

which indicates that lumping half of the total mass moment of inertia to each of the shaft ends results in an underestimation of the TNF.

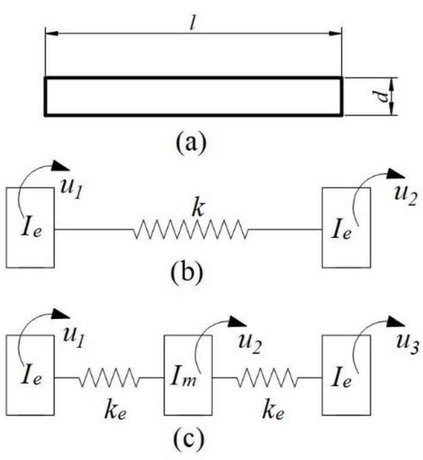

Fig. 1. (a) A shaft; (b) 2-DOF LPM; (c) 3-DOF system.

As the FE model will be used later, it is worthwhile determining the TNF of one-element FE model. For a single-element FE model, the stiffness matrix K and mass matrix M are given by [12]

$$K = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},\ M = \frac{I}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}. \quad (5)$$

It should be noted that the stiffness matrix of an FE model is the same as the one of the corresponding LPM while the mass matrices are different. Solving an eigenvalue problem with K and M yields

$$\omega_n = \frac{2\sqrt{3}}{\pi}\frac{\pi}{l}\sqrt{\frac{G}{\rho}} = 1.1025\omega_1 \quad (6)$$

which indicates that the one-element FE model results in an overestimation of the TNF. In general, to have an acceptable accuracy, the DOF of an FE model should be at least twice the number of TNFs desired.

To lump the mass moment of inertia properly, we define the following

$$I_e = aI \quad (7)$$

where a is a constant to be determined. Using (7) in (4) results in

$$\omega_n = \frac{\sqrt{2/a}}{\pi}\frac{\pi}{l}\sqrt{\frac{G}{\rho}} = \frac{\sqrt{2/a}}{\pi}\omega_1 \quad (8)$$

which yields

$$a = \frac{2}{\pi^2} \approx 0.20259. \quad (9)$$

This indicates that lumping about 20% of I on each of the shaft ends will preserve the 1st TNF.

Another choice is to put portion of I at one end and the rest of I at the other end, i.e.,

$$I_{el} = aI \text{ and } I_{er} = (1-a)I \quad (10)$$

where I_{el} and I_{er} are lumped mass moment of inertia at the left end and right end, respectively. In this case, the equivalent mass moment of inertia becomes

$$I_{eq} = \frac{I_{el}I_{er}}{I_{el} + I_{er}} = a(1-a)I. \quad (11)$$

Repeating the derivation procedure above yields

$$\frac{1}{a(1-a)} = \pi^2. \quad (12)$$

Solving (12) yields

$$a_{1,2} = 0.1144,\ 0.8856. \quad (13)$$

Note that $a_2 = 1 - a_1$. Distributing the mass moment of inertia based on (13) ensures the total mass moment of inertia unchanged. However, the LPM's nodal point is not in the middle of the two lumped mass moments of inertia, differing from the analytical solution. In what follows, wherever it is possible, the approximate lumped mass moments of inertia will be distributed evenly along the shaft or individual shaft segments.

Now we explore the possibility of using a 3-DOF LPM shown in Fig. 1(c) to approximate the shaft. We want to determine I_e and I_m so that the natural frequencies of the LPM match the first two TNFs of the shaft. The equations of motion for the 3-DOF LPM are defined by

$$\begin{aligned} \ddot{u}_2 - \ddot{u}_1 + k_e b(u_2 - u_1) - k_e c(u_3 - u_2) = 0 \\ \ddot{u}_3 - \ddot{u}_2 - k_e c(u_2 - u_1) + k_e b(u_3 - u_2) = 0 \end{aligned} \tag{14}$$

where

$$k_e = 2k,\ b = (I_e + I_m)/(I_e I_m),\ c = 1/I_m. \tag{15}$$

The natural frequencies can be found by solving the following eigenvalue problem

$$\det\begin{bmatrix} k_e b - \omega^2 & -k_e c \\ -k_e c & k_e b - \omega^2 \end{bmatrix} = \omega^4 - 2k_e b\omega^2 + k_e^2 b^2 - k_e^2 c^2 = 0 \tag{16}$$

Equating the roots of (16) to ω_1^2 and ω_2^2 where ω_1 and ω_2 are the first two TNFs defined in (1) yields

$$\begin{aligned} 2k_e b = \omega_1^2 + \omega_2^2 \\ k_e^2 b^2 - k_e^2 c^2 = \omega_1^2 \omega_2^2 \end{aligned} \tag{17}$$

Solving (17) for b and c and (15) for I_e and I_m yields

$$I_e = \frac{2k}{\omega_1^2},\ I_m = \frac{4k}{\omega_2^2 - \omega_1^2} \tag{18}$$

Consider a shaft with the following data: $d = 0.2$ m, $l = 1.05$ m, $G = 79.3 \times 10^3$ N/m², $\rho = 7.8 \times 10^3$ kg/m³ so that $k = 1.1863 \times 10^7$ Nm/rad, $J = 1.5708 \times 10^{-4}$ m⁴, $I = 1.2865$ kgm². If we approximate the shaft with a 2-DOF LPM, we have

$$I_e = 0.2607 \text{ m}^4.$$

If we approximate the shaft with a 3-DOF LPM, we have

$$\begin{aligned} I_e = 0.2607 \text{ m}^4,\ I_m = 0.1738 \text{ m}^4 \\ I_e/I = 0.20264,\ I_m/I = 0.13509 \end{aligned}$$

Note that both cases yield the same value for I_e.

III. NUMERICAL SOLUTION

The analysis above indicates that it is possible to have an analytical solution for an LPM to match up to two TNFs of a shaft with a constant diameter. When the number of TNFs is greater than 2, seeking an analytical solution becomes extremely involved and likely impossible. Also, a shaft with variable diameters does not have analytical solutions for its TNFs. In what follows, we develop a procedure to develop an optimal LMP based on nonlinear optimization techniques.

Start with a simple case shown in Fig. 1(c). Define the following

$$I_e = a_e I \text{ and } I_m = a_m I \tag{19}$$

where a_e and a_m are constant to be determined. The optimal values of a_e and a_m can be determined by minimizing the objective function defined by

$$E(\mathbf{x}) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\omega_i^a - \omega_i^t}{\omega_i^t}\right)^2} \tag{20}$$

where $\mathbf{x}$ is a vector of variables to be optimized, ω_i^t and ω_i^a are the i^th target TNF and approximate TNF, respectively, and N is the number of TNFs to be matched. This minimization problem can be solved by Matlab function fmincon which finds a constrained minimum of a function of several variables. For the problem under consideration, we need to specify initial values $\mathbf{x}_0$, lower bounds $\mathbf{x}_l$, upper bounds $\mathbf{x}_u$ of the variables. The function provides several algorithm options. In this study, the default SQP (sequential quadratic programming) algorithm is used.

First let's use the data given below (18) to test the proposed procedure. For this purpose, we choose $\mathbf{x}_0 = [0.2 \quad 0.2]$, $\mathbf{x}_l = [0 \quad 0]$, $\mathbf{x}_u = [0.5 \quad 0.5]$. The results obtained are $a_e^* = 0.20264$, $a_m^* = 0.13509$, $E = 1.3233 \times 10^{-8}$. A near zero objective function value indicates that the search converges to the analytical solution. By generating $\mathbf{x}_0$ randomly between 0 and 0.5, the minimization consistently converges to the same results, indicating that the global minimum is reached.

In what follows, we use several cases to test the proposed procedure. First consider approximating the shaft with a 4-DOF LPM shown in Fig. 2. In Fig. 2(a), the shaft is divided evenly so that $k = 3\,GJ/l$. In Fig. 2(b) the shaft is divided into $2l/7$, $3l/7$, $2l/7$ so that $k_1 = 7GJ/(2l)$ and $k_2 = 7GJ/(3l)$. As a 4-DOF LPM is used, the number of TNFs to be matched is 3 or $N = 3$ in (20). Using the same lower bounds and upper bounds for a_e, a_m, we run the Matlab script with randomly generated initial values between 0 and 0.5 multiple times. Table I lists the results obtained along with the true TNFs and TNFs from the solution of a 3-element FE model. In the table, Opt1 corresponds to the LPM in Fig. 2(a) with $a_e^* = 0.19286$ and $a_m^* = 0.11365$ while Opt2 corresponds to the LPM in Fig. 2(b) with $a_e^* = 0.19244$ and $a_m^* = 0.12090$. All the TNF values are given in kHz and RE represents the relative error in percentage between the true value and computed value. Comparing the three results shows that both Opt1 and Opt2 give more accurate results than the FE model and Opt1 is better than Opt2.

TABLE I. TNFS FOR THE MODELS IN FIG. 2.

	True	FE	RE	Opt1	RE	Opt2	RE
f_1	1.463	1.588	4.6	1.501	1.1	1.463	3.6
f_2	3.318	3.552	17.0	3.130	3.1	3.318	9.3
f_3	4.229	5.023	10.3	4.458	2.1	4.229	7.2
mean			10.6		2.1		6.7

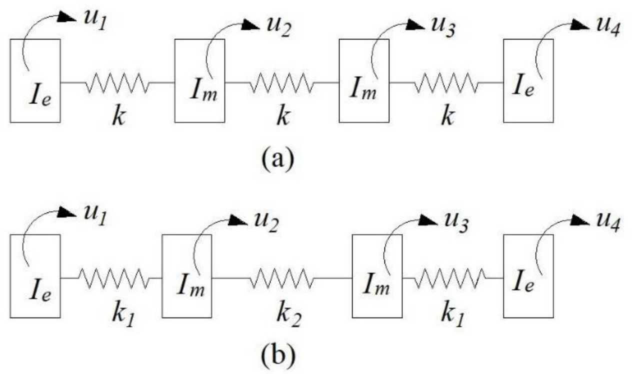

Fig. 2. (a) 4-DOF LPM with even division; (b) 4-DOF system with uneven division.

In many cases, it is not possible to relate the individual lumped mass moments of inertia to the total mass moment of inertia because a shaft is often divided into multi segments based on geometry. In this case, its LPM should be built based on these segments. The following case investigates how to tackle such a problem. Fig. 3 shows a shaft that is divided into 3 segments and its 10-DOF LPM.

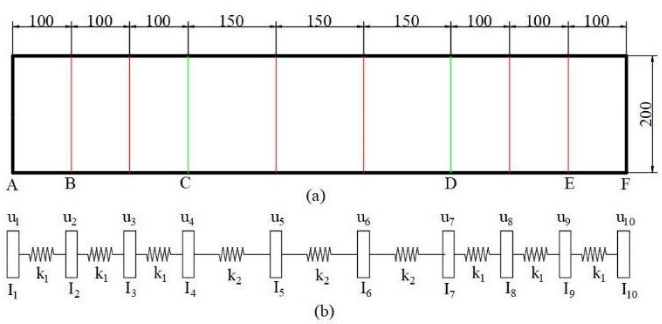

Fig. 3. (a) a shaft divided into 3 segments; (b) 10-DOF LPM.

We define the following:

$$\begin{aligned} &I_1 = I_{10} = a_e I_{AC}, I_2 = I_3 = I_8 = I_9 == a_m I_{AC} \\ &I_4 = I_7 = a_i\left(I_{AC} + I_{CD}\right), I_5 = I_6 = a_m I_{CD} \end{aligned} \quad (21)$$

where a_e, a_m, a_i are constant to be determined. The rational for the terms defined in (21) is that individual I is associated with its adjacent segment(s). Note that for this case, $k_1 = 1.2456 \times 10^8$ N.m/rad, $k_2 = 8.3043 \times 10^8$ N.m/rad and the number of TNFs to be matched is 9 or $N = 9$ in (20). Using the same lower bounds and upper bounds for a_e, a_m, a_i, we run the minimization script multiple times with the initial values generated randomly generated between 0 and 0.5. A set of the results corresponding to the smallest E value is listed in Table II along the analytical and FE values. The optimal constants are $a_e^* = 0.49364$, $a_m^* = 0.17864$, $a_i^* = 0.08918$. Comparison of the results indicates that the optimal LPM yields the TNFs that are quite close to the true ones. The accuracy of the optimal LMP is much better than the 9-element FE model. It is noted that $a_m^* \approx 2a_i^*$, indicating that the number of parameters to be optimized can be reduced to 2. To verify it, a 2-parameter optimization is conducted by replacing $I_4 = I_7 = a_i(I_{AC} + I_{CD})$ with $I_4 = I_7 = 0.5a_m(I_{AC} + I_{CD})$. The same result is obtained. Based on this case study, a general guideline for discretizing a shaft can be drawn. If possible, divide a shaft evenly. If a shaft must be divided into multiple segments, divide each of the segments evenly. Then define the following:

$$I_e = a_e I_s, I_m = a_m I_s, I_i = 0.5a_m\left(I_{ls} + I_{rs}\right) \quad (22)$$

where I_e and I_m are the lumped mass moments of inertia at the shaft's end and the location within a segment, respectively, I_i is the lumped mass moment of inertia at the location between two segments, I_s, I_{ls}, and I_{rs} are the mass moments of inertia of the segment, the left segment, and the right segment, respectively.

TABLE II. TNFS FOR THE MODELS IN FIG. 3.

	True	FE	RE	Opt1	RE
f_1	1.518	1.525	0.4	1.472	3.1
f_2	3.037	3.111	2.4	3.072	1.2
f_3	4.555	4.808	5.6	4.722	3.7
f_4	6.073	6.551	7.9	6.247	2.9
f_5	7.592	8.555	12.7	7.845	3.3
f_6	9.110	11.092	21.8	9.417	3.4
f_7	10.638	12.093	13.8	9.676	9.0
f_8	12.147	16.467	35.6	12.569	3.5
f_9	13.665	16.581	21.3	12.578	8.0
mean			13.5		4.2

The proposed procedure is further tested using a steel shaft with variable diameters shown in Fig. 4. The shaft is divided into 9 segments as shown in Fig. 4(a) so that it can be approximated by a 10-DOF LPM as shown in Fig. 4(b). There are 5 different polar moments of inertia given by

$$\begin{aligned} &J_1 = \pi(0.126)^4/32,\ J_2 = \pi(0.166)^4/32,\ J_3 = \pi(0.19)^4/32 \\ &J_4 = \pi(0.232)^4/32,\ J_5 = \pi(0.4)^4/32 \end{aligned}$$

Then the mass moments of inertia for the segments are

$$\begin{aligned} &I_{AB} = \rho(J_1 \times 0.1 + J_2 \times 0.05 + J_3 \times 0.0615 + J_4 \times 0.045) \\ &I_{BC} = \rho(J_3 \times 0.123 + J_5 \times 0.09) \\ &I_{EF} = \rho J_3 \times 0.155 \\ &I_{CD} = I_{DE} = I_{GH} = I_{HI} = I_{IJ} = I_{BC}, I_{FG} = I_{EF} \end{aligned}$$

The stiffnesses are given by

$$k_{11} = GJ_1 / 0.1,\ k_{12} = GJ_2 / 0.05$$
$$k_{13} = GJ_4 / 0.045,\ k_{14} = GJ_3 / 0.0615$$
$$k_1 = k_{11}k_{12}k_{13}k_{14} / \left(k_{12}k_{13}k_{14} + k_{11}k_{13}k_{14} + k_{11}k_{12}k_{14} + k_{11}k_{12}k_{13}\right)$$
$$k_{21} = GJ_5 / 0.0615,\ k_{22} = GJ_5 / 0.09,\ k_{23} = k_{21}$$
$$k_2 = k_3 = k_4 = k_7 = k_8 = k_9 =$$
$$k_{21}k_{22}k_{23} / \left(k_{22}k_{23} + k_{21}k_{23} + k_{21}k_{22}\right)$$
$$k_5 = k_6 = GJ_3 / 0.31$$

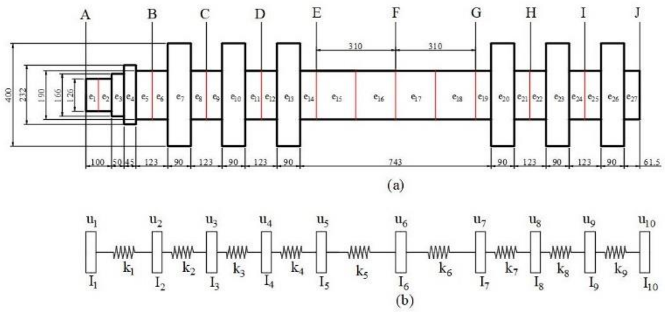

Fig. 4. (a) a shaft with variable diameters; (b) 10-DOF LPM.

For the sake of comparison, an LPM named LPM1 is established by simply concentrating half of the mass moment of inertia of each of the segments to its ends

$$\begin{aligned} &I_1 = 0.5 I_{AB},\ I_2 = 0.5\left(I_{AB} + I_{BC}\right) \\ &I_3 = I_4 = I_8 = I_9 = 0.5\left(I_{BC} + I_{CD}\right) \\ &I_5 = I_7 = 0.5\left(I_{DE} + I_{EF}\right),\ I_6 = 0.5\left(I_{EF} + I_{FG}\right),\ I_{10} = 0.5 I_{IJ} \end{aligned} \tag{23}$$

For the optimal LPM named LPM2, define the lumped mass moments of inertia as follows

$$\begin{aligned} &I_1 = a_1 I_{AB},\ I_2 = a_2 I_{AB} + a_3 I_{BC} \\ &I_3 = I_4 = I_8 = I_9 = a_4\left(I_{BC} + I_{CD}\right) \\ &I_5 = I_7 = a_5 I_{DE} + a_6 I_{EF} \\ &I_6 = a_7\left(I_{EF} + I_{FG}\right),\ I_{10} = a_8 I_{IJ} \end{aligned} \tag{24}$$

where a_i, $i = 1,2,\cdots,8$, are constant to be determined. As there is no analytical solution for TNFs, a 27-element FE model is used to determine TNFs. The first 9 TNFs are to be matched using the optimal LPM. Setting the lower bound as 0 and the upper bound as 4, we run the minimization script multiple times using the initial values generated randomly between 0 and 4. The optimal parameters corresponding to the smallest E value are given below

$$a_1^* = 1.9379,\ a_2^* = 0.2778,\ a_3^* = 0.0329,\ a_4^* = 0.2705$$
$$a_5^* = 0.0589,\ a_6^* = 0.5998,\ a_7^* = 0.1087,\ a_8^* = 2.5913$$

Table III compares the results. As shown in the table, the TNFs from the optimal LPM are very close to those from the FE model. It is also noted that the LMP1 can give satisfactory approximation only up to the 3rd mode. To further reduce the number of parameters, we define LPM3 as

$$\begin{aligned} &I_1 = a_1 I_{AB},\ I_2 = a_2\left(I_{AB} + I_{BC}\right), \\ &I_3 = I_4 = I_8 = I_9 = a_3\left(I_{BC} + I_{CD}\right) \\ &I_5 = I_7 = a_4\left(I_{DE} + I_{EF}\right),\ I_6 = a_5\left(I_{EF} + I_{FG}\right),\ I_{10} = a_6 I_{IJ}. \end{aligned} \tag{25}$$

The optimal parameters obtained are

$$a_1^* = 2.0786,\ a_2^* = 3.9997,\ a_3^* = 0.2511$$
$$a_4^* = 0.0658,\ a_5^* = 0.1148,\ a_6^* = 0.1332.$$

As shown in Table III, the accuracy of LPM3 is slightly better than that of LPM2.

TABLE III. TNFS FOR THE MODELS IN FIG. 4.

	FE	LMP1	RE	LMP2	RE	LMP3	RE
f_1	0.3368	0.3143	6.7	0.3433	1.8	0.3485	3.6
f_2	0.9360	0.9738	4.0	0.9330	0.3	0.9380	0.2
f_3	1.0983	1.1188	1.9	1.0742	2.2	1.0660	2.9
f_4	1.8222	1.6405	10.0	1.9238	5.6	1.8634	2.3
f_5	2.3799	1.7961	24.5	2.2530	5.3	2.3456	1.4
f_6	3.3209	1.9535	41.2	3.3794	1.8	3.2827	1.2
f_7	4.0178	2.0709	48.5	3.9556	1.6	3.8871	3.3
f_8	4.6956	2.1522	54.2	4.6768	0.4	4.7848	1.9
f_9	7.3845	3.4171	53.7	7.3692	0.2	7.4364	0.7
mean			27.2		2.1		1.9

IV. CONCLUSION

In this study, the development of optimal lumped-parameter models (LPMs) for the torsional vibration analysis of shafts was investigated. Beginning with a shaft of constant diameter, an analytical approach was employed to construct a two-degree-of-freedom (2-DOF) LPM capable of matching the first two torsional natural frequencies (TNFs) exactly. To address shafts with varying diameters and to match more than two TNFs, a numerical procedure based on nonlinear optimization techniques was developed. Through several case studies, the proposed method was shown to generate multi-DOF LPMs that accurately replicate the lower-order TNFs of shafts with complex geometries.

The developed modeling approach has direct applications in the design and analysis of rotating machinery where torsional vibration is critical—such as crankshafts in reciprocating compressors, drive shafts in marine propulsion systems, and transmission shafts in power generation equipment. By ensuring that the key TNFs are well-captured, the model aids in avoiding resonance conditions and improving the reliability of these systems.

To validate and verify the developed model, several approaches can be considered. For analytical and numerical verification, the natural frequencies obtained from the LPM can be compared with those computed via high-fidelity finite element (FE) models. For experimental validation, the natural frequencies and dynamic responses of a physical shaft system can be measured using a torsional vibration test rig. A prototype of such a testing platform is currently under development and will be used in future work to assess the accuracy of the optimal LPMs in predicting real-world torsional behavior.

Acknowledgment

This research was funded by the Natural Sciences and Engineering Research Council of Canada, Alliance Grants (ALLRP 581073-22).

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